

Lectures on Particle Physics

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Part I

Preliminaries: Quantum Field Theory

Chapter 1 What is a Quantum Field Theory?

Our aim in this course is to describe elementary particles. This description must necessarily marry Quantum Mechanics (because particles are subatomic objects) and Special Relativity (because, given their small masses, elementary particles are easily relativistic).

1.1 Why the Schroedinger equation cannot work?

Remember that the Schroedinger equation in the abstract Hilbert space \mathcal{H} is given by

$$i\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle, \quad (1.1)$$

where H is the Hamiltonian operator. Notice that we set $\hbar = 1$ because we will work most of the times in natural units, in which $\hbar = 1 = c$. For a free particle in position space this can be written as

$$i\frac{\partial\psi(t, \mathbf{x})}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(t, \mathbf{x}), \quad (1.2)$$

where $\psi(t, \mathbf{x}) \equiv \langle \mathbf{x}|\psi(t)\rangle$ is the wave function. As we can see, time appears via a first derivative, while space coordinates appear through second derivatives. Remembering that Lorentz transformations mix time and space components, it is immediate to see that the Schroedinger equation cannot be relativistically invariant. This is already a strong indication that we will need to generalize Eq. (1.2) to a wave equation that is relativistically invariant. A second problem emerges when marrying Quantum Mechanics with special Relativity, which we will call the *localization problem*. The argument (due to Bohr), goes as follows: consider a closed box containing one particle. Suppose one of the walls of the box is mobile, in such a way that we can diminish the volume in which the particle can move. This means that we can in principle localize the particle as much as we want. We know however that quantum effects are important on scales of the order of the Compton wavelength associated with the particle,

$$\lambda_C = \frac{h}{mc} = \frac{2\pi}{mc}. \quad (1.3)$$

When the available volume inside the box becomes of the order $V \sim \lambda_C^3 \sim 1/m^3$, the uncertainty principle tells us that

$$pV^{1/3} \gtrsim 1 \quad \Rightarrow \quad p \gtrsim m. \quad (1.4)$$

This means that localization in a relativistic theory implies that we have enough energy to produce pairs, and our one-particle-quantum-mechanics theory ceases to be valid. We thus conclude that

A relativistic quantum theory must be a many particle theory.

A quantum many body theory is what we call *Quantum Field Theory* (QFT).

1.2 How to construct a QFT

We now sketch the steps behind the construction of a QFT. Of course we will not be able to enter into too many details, since the topic is very vast and a dedicated course is needed to appreciate all the details. We will describe the general ideas behind the construction of QFTs focussing only on the aspects that will be more relevant for us.

1.2.1 Creation and annihilation operators

The central object in the construction of a QFT is the notion of creation and annihilation operators. We will start from a *vacuum state* without particles, $|0\rangle$, and define creation and annihilation operators according to

$$a_p^\dagger |0\rangle = |p\rangle, \quad a_p |0\rangle = 0. \quad (1.5)$$

Here p is a multiindex label that can be written as

$$p = \{\mathbf{p}, \lambda_p\}, \quad (1.6)$$

where \mathbf{p} is the particle 3-momentum and λ_p represent all other quantum numbers (spin, charges, etc) needed to define the state of the particle. We focus for the moment on creation and annihilation operators that create particles of definite momentum. Multi-particle states are constructed from the vacuum by repeated application of the creation operators according to

$$|p_1, \dots, p_N\rangle = a_{p_1}^\dagger \dots a_{p_N}^\dagger |0\rangle. \quad (1.7)$$

As we know from the quantum mechanics courses, particles are either *bosons* or *fermions*. Bosons have states which are symmetric under exchange of quantum numbers, while fermions have states which are antisymmetric. This property can be easily implemented in the formalism requiring

$$[a_p, a_q^\dagger] = \delta_{\lambda_p \lambda_q} \delta^3(p - q) \quad (1.8)$$

for bosons and

$$\{a_p, a_q^\dagger\} = \delta_{\lambda_p \lambda_q} \delta^3(p - q) \quad (1.9)$$

for fermions.

It is also easy to see that the operator

$$\mathcal{N}_p \equiv a_p^\dagger a_p \quad (1.10)$$

counts the number of particles in the state with quantum numbers p , while the Hamiltonian operator is given by

$$H = \sum_p E_p a_p^\dagger a_p, \quad E_p = \begin{cases} \frac{p^2}{2M} \\ \sqrt{p^2 + M^2} \end{cases}, \quad (1.11)$$

where M is the mass of the particle.

1.2.2 The cluster decomposition

One of the properties we want to include in our multiparticle theory is the so-called *cluster decomposition*. What this means can be understood as follows: take two very far experiments, one located at point \mathbf{x} and the other at point \mathbf{y} . Unless we concoct things in such a way to create a correlated state at the beginning of the experiment, what will happen in one experiment should be independent from what happens in the other experiment. This very physical and intuitive requirement goes under the name of cluster decomposition. What does it imply? From the practical point of view it implies that the probability for some event A to happen in the apparatus located at \mathbf{x} and the probability for some other event B to happen in the apparatus located at \mathbf{y} should be multiplied to compute the total probability,

$$P = P_A(\mathbf{x})P_B(\mathbf{y}). \quad (1.12)$$

In quantum mechanics this means that the amplitude for the events to happen should factorize as

$$\mathcal{A} = \mathcal{A}_A(\mathbf{x})\mathcal{A}_B(\mathbf{y}), \quad (1.13)$$

and that the time evolution of the experiment located at \mathbf{x} should have nothing to do with the time evolution of the experiment located at \mathbf{y} . Since time evolution is given by the exponential of the Hamiltonian operator, the cluster decomposition implies that we should be able to schematically write the time evolution operator as

$$U = \prod_{\mathbf{x}} U(\mathbf{x}), \quad (1.14)$$

where $U(\mathbf{x})$ represents the time evolution of the events happening at \mathbf{x} . This can be automatically if the Hamiltonian can be written in terms of an Hamiltonian density as

$$H = \int d^3x \mathcal{H}(\mathbf{x}). \quad (1.15)$$

Indeed, in this case we have schematically

$$U = e^{-iHt} = e^{-i \int d^3x \mathcal{H}(\mathbf{x})t} = e^{-i \sum \mathcal{H}(\mathbf{x})t} = \prod_{\mathbf{x}} e^{-i\mathcal{H}(\mathbf{x})t} = \prod_{\mathbf{x}} U(\mathbf{x}) \quad (1.16)$$

which is the required factorization. Notice that this is a new element in the quantum theory of many bodies which is not present in one-particle QM: interactions are *local*, i.e. they depend on the position.

1.2.3 Field operators

As we just saw, the cluster decomposition forces us to write the Hamiltonian in terms of positions variables \mathbf{x} . Clearly, this feature is not present in the Hamiltonian of Eq. (1.11). How should we then proceed? The solution is to introduce *field operators*, i.e. the analog of the creation and annihilation operators in position space. More specifically, we introduce two operators

$$\psi(\mathbf{x}) = \sum_p u_p(\mathbf{x}) a_p, \quad \psi^\dagger(\mathbf{x}) = \sum_p u_p^*(\mathbf{x}) a_p^\dagger. \quad (1.17)$$

We will specify the properties of the $u_p(\mathbf{x})$ functions in a moment. The utility of such operators comes from the fact that they can be used to write the Hamiltonian density $\mathcal{H}(\mathbf{x})$.

Let us start discussing first the non-relativistic case. If we choose $u_p(\mathbf{x})$ to be a solution of the time-independent wave equation

$$\left(\frac{-\nabla^2}{2m} + V(\mathbf{x}) \right) u_p(\mathbf{x}) = \left(\frac{p^2}{2m} + V(\mathbf{x}) \right) u_p(\mathbf{x}), \quad (1.18)$$

then it can be easily shown that

$$\mathcal{H}(\mathbf{x}) = \psi^\dagger(\mathbf{x}) \left(\frac{-\nabla^2}{2m} + V(\mathbf{x}) \right) \psi(\mathbf{x}), \quad (1.19)$$

i.e. we are able to implement the cluster decomposition. It is interesting to observe that, using the fact that the functions $u_p(\mathbf{x})$ is a solution of the Schrodinger equation (i.e. they form a complete set of orthogonal functions), we obtain

$$[\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})] = \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad \{\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\} = \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad (1.20)$$

where the first expression is valid for bosons and the second for fermions. We see that the field operators behave like creation and annihilation operators in position space.

We now introduce time dependence in the picture constructed so far. We can easily pass to Heisenberg picture remembering that

$$\dot{a}_p = i [H, a_p], \quad (1.21)$$

where the Hamiltonian is given in Eq. (1.11). We immediately obtain

$$\begin{aligned}
 \dot{a}_p &= \sum_q iE_q [a_q^\dagger a_q, a_p] \\
 &= \sum_q iE_q \left(a_q^\dagger [a_q, a_p] + [a_q^\dagger, a_p] a_q \right) \\
 &= \sum_q iE_q \left(-\delta_{\lambda_p \lambda_q} \delta^3(\mathbf{p} - \mathbf{q}) \right) a_q \\
 &= -iE_p a_p .
 \end{aligned} \tag{1.22}$$

This equation can be immediately integrated to obtain

$$a_p(t) = a_p(0)e^{-iE_p t} . \tag{1.23}$$

This result can be translated at the level of field operator:

$$\psi_m(\mathbf{x}, t) = U^\dagger(t) \left(\sum_p u_p(\mathbf{x}) a_p \right) U(t) = \sum_p u_p(\mathbf{x}) e^{-iE_p t} a_p = \sum_p u_p(\mathbf{x}, t) a_m(p) . \tag{1.24}$$

We see that the time dependence can be simply implemented promoting the solution of the time-independent Schrodinger equation to a solution of the time-dependent Schrodinger equation. The immediate consequence is that the field operator $\psi(\mathbf{x}, t)$ satisfies the wave equation

$$\left[i \frac{\partial}{\partial t} - \left(\frac{-\nabla^2}{2m} + V(\mathbf{x}) \right) \right] \psi(\mathbf{x}, t) = 0 \tag{1.25}$$

simply because the functions $u_p(\mathbf{x}, t)$ satisfies it.

We conclude with an interesting observation. We know that classical systems can be quantized using the procedure known as canonical quantization (apart from some ordering problems which will not interest us now). We have shown in Eq. (1.25) that the field *operator* satisfies the Schrodinger equation. The question we set up to answer is: can this result be obtained from canonically quantize the theory of a *classical* Schrodinger field? The answer is yes, as we are now going to see. Take a classical field $\psi(x)$. The Schrodinger equation is obtained from an action of the form

$$\mathcal{S} = \int d^4x \psi^*(x) \left[i \frac{\partial}{\partial t} - \left(\frac{-\nabla^2}{2m} + V(\mathbf{x}) \right) \right] \psi(x) \tag{1.26}$$

applying the usual Euler-Lagrange equations of motion. As we know from Quantum Mechanics, canonical quantization amounts to promote classical variables (the field, in this case) to operators, with Poisson brackets giving the commutator according to

$$\{ \cdot, \cdot \} \rightarrow i [\cdot, \cdot] . \tag{1.27}$$

Let us apply this procedure to the Schrodinger field. The only non-trivial Poisson bracket

is between the field and its conjugate momentum

$$\pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\psi}} = i\psi^*(x) . \quad (1.28)$$

The Poisson brackets are given by

$$\begin{aligned} \{\psi(\mathbf{x}, t), \pi(\mathbf{y}, t)\} &= \int d^3z \left(\frac{\partial \psi(\mathbf{x}, t)}{\partial \psi(\mathbf{z}, t)} \frac{\partial \pi(\mathbf{y}, t)}{\partial \pi(\mathbf{z}, t)} - \frac{\partial \psi(\mathbf{x}, t)}{\partial \pi(\mathbf{z}, t)} \frac{\partial \pi(\mathbf{y}, t)}{\partial \psi(\mathbf{z}, t)} \right) \\ &= \delta^{(3)}(\mathbf{x} - \mathbf{y}) . \end{aligned} \quad (1.29)$$

This means that the commutator between the *quantum* Schrodinger field and its conjugate momentum gives

$$[\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{y}, t)] = \delta^{(3)}(\mathbf{x} - \mathbf{y}) , \quad (1.30)$$

which is exactly the result expected from the procedure outline above. We thus obtain a very important result: a classical field theory quantized according to canonical quantization reproduces the results of a many-body theory.

1.2.4 Relativistic wave equations

We have seen in the previous section that the field operator $\psi(\mathbf{x}, t)$ satisfies the wave equation. What are the wave equations that relativistic fields should satisfy? The answer is well-known from the study of the Poincaré group. They are

$$\begin{aligned} \text{(Klein - Gordon)} \quad & (\square + m^2)\phi(x) = 0 , \\ \text{(Dirac)} \quad & (\not{\partial} - m)\psi(x) = 0 , \\ \text{(Proca)} \quad & \partial^\mu V_{\mu\nu}(x) + m^2 V_\nu(x) = 0 , \end{aligned} \quad (1.31)$$

where in the Proca equation we have defined $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. It can be shown that these are the most general differential equations up to second order derivatives that are covariant under Lorentz transformations. It can easily be shown that all the wave equations imply that the fields must satisfy the Klein-Gordon (KG) equation, with the other equations (like the Dirac or Proca one) imposing additional constraints on the solutions. We will write our fields as

$$\varphi_A(x) = \int d^3p \sum_\lambda u_A^\lambda(p) \varphi_{\text{KG}}(p, x) a_p^\lambda, \quad (1.32)$$

where A is a Lorentz index (for instance, a spinor index in the case of the Dirac spinor, or the usual Lorentz index μ in the case of the Proca field). The index λ refers now to the spin (or helicity, for massless particles) degrees of freedom. $\varphi_{\text{KG}}(p, x)$ is the solution of the KG equation. Since this equation is a wave equation, it admits plane wave solutions: decomposing

$$\varphi_{\text{KG}} = \varphi_0(t) e^{i\mathbf{p}\mathbf{x}} \quad (1.33)$$

and defining $E_p = \sqrt{\mathbf{p}^2 + M^2}$, we obtain that the time dependent solution must obey

$$\ddot{\varphi}_0 + E_p^2 \varphi_0 = 0, \quad \Rightarrow \quad \varphi_0^\pm(t) = \varphi_0(0) e^{\mp i E_p t}, \quad (1.34)$$

where the superscript (\pm) refers to positive and negative energy states, respectively. With a redefinition of the field we can always fix $\varphi_0(0) = 1$. We are thus forced to introduce two types of fields: one with positive energy, the other with negative energy. Although not clear right now, the negative-energy fields will play a fundamental role in guaranteeing that causality is satisfied in the theory.

Putting all together, the relativistic quantum fields obeying the wave equations in Eq. (1.31) are

$$\begin{aligned} \text{(Klein - Gordon)} \quad \phi^{(\pm)}(x) &= \int d^3p e^{i(\mp E_p t + \mathbf{p} \cdot \mathbf{x})} a_p^\pm, \\ \text{(Dirac)} \quad \psi^{(\pm)}(x) &= \int d^3p \sum_\lambda u^{\lambda, \pm}(p) e^{i(\mp E_p t + \mathbf{p} \cdot \mathbf{x})} a_p^{\lambda, \pm}, \\ \text{(Proca)} \quad V_\mu^{(\pm)} &= \int d^3p \sum_\lambda \epsilon_\mu^{\lambda, \pm}(p) e^{i(\mp E_p t + \mathbf{p} \cdot \mathbf{x})} a_p^{\lambda, \pm}, \end{aligned} \quad (1.35)$$

where $u(p)$ and $\epsilon_\mu(p)$ satisfy the Dirac and Proca equations, respectively. To simplify the notation, we notice that the exponent in the exponential of the positive-energy solutions can be written as $-i(E_p t - \mathbf{p} \cdot \mathbf{x}) = -ipx$ in a relativistic invariant way. As for the negative-energy solutions, we can always redefine $\mathbf{p} \rightarrow -\mathbf{p}$ in the integral, obtaining $+ipx$ in the exponent. Redefining also

$$u^{\lambda, +}(p) = u_\lambda(p), \quad u^{\lambda, -}(-p) = v_\lambda(p), \quad \epsilon_\mu^{\lambda, +}(p) = \epsilon_\mu^\lambda(p), \quad \epsilon_\mu^{\lambda, -}(-p) = \bar{\epsilon}_\mu^\lambda(p), \quad (1.36)$$

as well as

$$a^{\lambda, +}(p) = a_\lambda(p), \quad a^{\lambda, -}(-p) = b_\lambda^\dagger(p), \quad (1.37)$$

we can finally write

$$\begin{aligned} \phi^{(+)}(x) &= \int d^3p e^{-ipx} a(p), & \phi^{(-)}(x) &= \int d^3p e^{ipx} b^\dagger(p), \\ \psi^{(+)}(x) &= \int d^3p \sum_\lambda u_\lambda(p) e^{-ipx} a_\lambda(p), & \psi^{(-)}(x) &= \int d^3p \sum_\lambda v_\lambda(p) e^{ipx} b_\lambda^\dagger(p), \\ V_\mu^{(+)}(x) &= \int d^3p \sum_\lambda \epsilon_\mu^\lambda(p) e^{-ipx} a_\lambda(p), & V_\mu^{(-)}(x) &= \int d^3p \sum_\lambda \bar{\epsilon}_\mu^\lambda(p) e^{ipx} b_\lambda^\dagger(p). \end{aligned} \quad (1.38)$$

The functions $u(p)$, $v(p)$, $\epsilon_\mu(p)$ and $\bar{\epsilon}_\mu(p)$ obey the Dirac and Proca equations in momentum space:

$$\begin{aligned} (\not{p} - M)u(p) &= 0, & (\not{p} + M)v(p) &= 0, \\ p_\mu \epsilon^\mu(p) &= 0, & p_\mu \bar{\epsilon}^\mu(p) &= 0. \end{aligned} \quad (1.39)$$

Notice that each one of the solutions are necessarily complex. We have explicitly written

the negative energy solution in terms of a new creation operator $b^\dagger(p)$. As we are going to see in the next chapter, this operator creates antiparticles, and is connected to the possible charges that the field could carry. If a particle is its own antiparticle then we can take $b(p) = a(p)$.

The physical meaning of the negative-energy solutions is still quite unclear. Can we disregard them completely? The answer is no. The reason is twofold:

- The first problem is a problem of **measurability**. Take the Hamiltonian density $\mathcal{H}(\mathbf{x})$. Since the Hamiltonian must be an observable, it must be hermitian. However, none of the positive-energy solutions in Eq. (1.35) are hermitian. This already implies that $\mathcal{H}(\mathbf{x})$ cannot be constructed out of the positive-energy solutions only, but both positive-energy and negative-energy solutions must be present;
- The second problem is a problem of **causality**. Take two points \mathbf{x} and \mathbf{y} at a space-like separation. Since no signal can be exchanged between the two, it must be possible to measure $\mathcal{H}(\mathbf{x})$ and $\mathcal{H}(\mathbf{y})$ simultaneously without changing the result of the other measurement. In other words, $\mathcal{H}(\mathbf{x})$ and $\mathcal{H}(\mathbf{y})$ must be compatible operators,

$$[\mathcal{H}(\mathbf{x}), \mathcal{H}(\mathbf{y})] = 0 \quad (\mathbf{x} - \mathbf{y})^2 < 0. \quad (1.40)$$

It turns out that both points above are solved if our quantum fields are constructed as

$$\varphi_A(x) = \varphi_A^{(+)}(x) + \varphi_A^{(-)}(x), \quad (1.41)$$

i.e. the positive and negative energy solutions must enter on the same foot in the quantum field. We will study in detail the physical interpretation of the negative-energy solutions in Section 2.2. ¹

Let us conclude this section with a comment on the normalization of the fields. The measure d^3p in Eq. (1.35) is not Lorentz invariant, and the same is true for the commutation/anticommutation relations of Eq. (1.30). The Lorentz invariant combinations are given by

$$\frac{d^3p}{2E_p}, \quad E_p \delta^{(3)}(\mathbf{p} - \mathbf{q}). \quad (1.42)$$

This suggests that we write our field operators as integrals over $d^3p/(2E_p)$, using creation operators

$$\alpha_m(p) \equiv (2\pi)^{3/2} \sqrt{2E_p} a_m(p), \quad (1.43)$$

where the numerical factor $(2\pi)^{3/2}$ is inserted for later convenience. Putting all together

¹Another information that can be extracted from the fact that the positive and negative-energy solutions must enter on equal foot is the *spin-statistic* theorem: bosons can only have integer spin, while fermions can only have semi-integer spin.

we obtain that the typical relativistic quantum field can be written as

$$\varphi_A(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_{\lambda} \left(u_A^{\lambda}(p) \alpha_{\lambda}(p) e^{-ipx} + v_A^{\lambda}(p) \beta_{\lambda}^{\dagger}(p) e^{ipx} \right). \quad (1.44)$$

For real fields we can take $\beta_{\lambda}^{\dagger}(p) = \alpha_{\lambda}^{\dagger}(p)$ and $v_A^{\lambda}(p) = u_A^{\lambda*}(p)$.

1.3 Lagrangians for free particles

As it happens in the case of the Schrodinger equation, we can obtain the same results starting from a classical field theory and applying canonical quantization. A classical field theory is defined in terms of a Lagrangian (density) which must be a scalar. This forces us to consider only Lagrangians that are Lorentz invariant. The only case in which this will not be completely straightforward is when massless spin 1 particles are involved. We will study the subtleties of this case in next section. In this section we just show the Lagrangian for spin 0, 1/2 and 1 free particles. They are

$$\begin{aligned} \mathcal{L}_{\text{scalar}} &= \partial_{\mu} \phi^{\dagger} \partial_{\mu} \phi - m^2 \phi^{\dagger} \phi, \\ \mathcal{L}_{\text{real scalar}} &= \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{1}{2} m^2 \phi^2, \\ \mathcal{L}_{\text{Dirac}} &= \bar{\psi} (i \not{\partial} - m) \psi, \\ \mathcal{L}_{\text{vector}} &= -\frac{1}{2} V_{\mu\nu}^{\dagger} V_{\mu\nu} + m^2 V_{\mu}^{\dagger} V_{\mu}, \\ \mathcal{L}_{\text{real vector}} &= -\frac{1}{4} (V_{\mu\nu})^2 + \frac{m^2}{2} (V_{\mu})^2. \end{aligned} \quad (1.45)$$

We present in Appendix A the details of the computation.

1.4 Massive and massless spin 1 particles

Let us conclude this chapter pointing out a crucial point for all our subsequent discussion. In Eq. (1.44) we have explicitly written a sum over polarizations m , without entering into any detail. We now discuss this point, since it plays an essential role in the physics of photons.

We start by counting the number of (spin) degrees of freedom of a **massive** particle. A particle of spin 0 has $2j + 1 = 1$ degree of freedom. A particle of spin 1/2 has 2 degrees of freedom. A particle of spin 1 has 3 degrees of freedom. This means that the sum over polarizations must be taken over 1, 2 and 3 states, respectively.

Let us focus on the case of spin 1, which will be the one for which the massless limit will prove more subtle. In this case we need 3 polarization vectors $\epsilon_{\mu}^m(p)$. To understand what form they take, it is useful to go back to the Proca equation in Eq. (1.31). Applying

∂^ν and remembering the antisymmetry of $V_{\mu\nu}$ we obtain the constraint $\partial^\mu V_\mu = 0$. At the level of polarization vectors this means that each polarization must satisfy $p^\mu \epsilon_\mu^m(p) = 0$. We now use the so-called method of induced representations of the Lorentz group. It amounts to the following observations:

- To study the Lorentz transformations of objects that depend on the momentum (like our polarization vectors), we start by choosing a reference momentum k^μ in which the description of the physics is particularly simple. We then write

$$p = L_{k \rightarrow p} k , \quad (1.46)$$

where $L_{k \rightarrow p}$ is an appropriate Lorentz transformation. We will see shortly that this matrix is not uniquely defined, but can be multiplied on the right by any Lorentz transformation that leaves k invariant. If we now apply a second Lorentz transformation Λ to p , we can write

$$\Lambda p = \Lambda L_{k \rightarrow p} k = L_{k \rightarrow \Lambda p} k = L_{k \rightarrow \Lambda p} M k \quad (1.47)$$

where in the last step we have introduced the matrix M belonging to the *little group* of k , defined as the group of Lorentz transformations that leave k invariant, $Mk = k$. This means that to study any Lorentz transformation Λ we can first study the little group, and then boost using $L_{k \rightarrow \Lambda p}$;

- In the case of our polarization vectors we define

$$\epsilon_m^\mu(p) = L_{k \rightarrow p} \epsilon_m^\mu(k) ; \quad (1.48)$$

- For a massive particle we can choose k as the rest frame momentum, $k = (m, \mathbf{0})$. In this frame we can always choose the three polarization vectors according to

$$\epsilon_\pm^\mu(k) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0) , \quad \epsilon_L^\mu = (0, 0, 0, 1) . \quad (1.49)$$

Notice that this is a complete basis when restricted to the 3-dimensional space. The little group in this case is simply the rotation group $SO(3)$, in such a way that the only effect of a little group transformation on the polarization vectors is to mix them one with the other. This means that a Lorentz transformation on the polarization vectors amounts to

$$\Lambda \epsilon_m^\mu(p) = \Lambda L_{k \rightarrow p} \epsilon_m^\mu(k) = L_{k \rightarrow \Lambda p} \epsilon_m^\mu(k) , \quad (1.50)$$

i.e. they transform as usual 4 vectors.

We now turn to the case of a massless vector. In this case the reference momentum k can at most be chosen as $k = (E, 0, 0, E)$, and only two polarization vectors are needed, which we can choose as

$$\epsilon_\pm^\mu(k) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0) , \quad (1.51)$$

exactly as in the massive case. Now comes the catch. Let us repeat the reasoning above and inspect what happens with a transformation of the little group of k . Among the transformations of the little group there are rotations around the z axis that simply mix the polarization vectors among each other. There are, however, other elements in the little group whose action is different. An example is given by the Lorentz transformation

$$M = \begin{pmatrix} 3/2 & 1 & 0 & -1/2 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1/2 & 1 & 0 & 1/2 \end{pmatrix}. \quad (1.52)$$

It is immediate to see that $Mk = k$, as it should be for an element of the little group. When we apply it to $\epsilon_{\pm}^{\mu}(k)$ we obtain

$$M\epsilon_{\pm}(k) = \epsilon_{\pm}(k) + \frac{1}{\sqrt{2}E}k. \quad (1.53)$$

This means that *there are transformation in the little group that not only do not leave the polarization invariant, but that they shift them in the direction of the momentum k* . It can be shown that this is a general result. But then applying a generic Lorentz transformation we obtain

$$\Lambda\epsilon_{\pm}(p) = L_{k \rightarrow \Lambda p} M \epsilon_{\pm}(k) = L_{k \rightarrow \Lambda p} (\epsilon_{\pm}(k) + \alpha k) = \epsilon_{\pm}(\Lambda p) + \alpha \Lambda p, \quad (1.54)$$

where α is some constant. The main result is clear: *when we embed a massless spin 1 particle in a vector field, the vector field does not transform as a 4-vector but it shift in momentum space by a term proportional to the momentum itself*. In position space this means that a Lorentz transformation amounts to

$$V^{\mu}(x) \rightarrow \Lambda^{\mu}_{\nu} V^{\nu}(x) + \partial^{\mu} \xi(x), \quad (1.55)$$

since the shift proportional to the derivative amounts to a shift proportional to the momentum. Notice that this transformation contains what is usually called “gauge transformation”. We have recovered it here as a consequence of the redundancy of our description (i.e. of the fact that we insist in embedding two degrees of freedom in a 4-component object). Let us stress the last point, since it is often a confusing point in the literature: Eq. (3.6) is not a symmetry transformation, rather the statement that when we want to describe a massless vector in terms of a 4-component object $V_{\mu}(x)$ we need to admit that all the configurations that differ by a derivative are equivalent, i.e.

$$V_{\mu}(x) \sim V_{\mu}(x) + \partial_{\mu} \xi(x). \quad (1.56)$$

In this sense, a gauge transformation is simply an expression of a redundancy and **not a symmetry transformation**.

1.5 Additional reading

- S.Weinberg, “The Quantum Theory of Fields” vol. 1;
- M.Peskin and Daniel Schroeder, “An Introduction To Quantum Field Theory”;
- S.Coleman, “Lectures on Quantum Field Theory”;
- M.Schwartz, “Quantum Field Theory and the Standard Model”.

Chapter 2 Interactions in Quantum Field Theory

In the previous chapter we have focused on free particles, i.e. on Lagrangians that are at most quadratic in the fields. In this lecture we will introduce interactions, defined as terms which are of cubic or higher order in \mathcal{L} . For instance, interactions will be of the form ϕ^3 , ϕ^4 , $\phi\psi\psi$, or any other combination which is allowed by (i) Lorentz invariance, (ii) symmetries, (iii) gauge invariance (if needed).

We will first explore the consequences of gauge invariance, and then turn on a review of symmetries.

2.1 Consequences of gauge invariance

As we have seen, the mismatch between the number of degrees of freedom in a massless vector A_μ and its physical (2) dof forces us to introduce an equivalence relation that amounts to state that the physics cannot change in the presence of the longitudinal degree of freedom,

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \xi(x) . \tag{2.1}$$

The simplest coupling between A_μ and other particles, requiring Lorentz invariance, is of the form

$$\mathcal{L}_{int} = g A_\mu J^\mu , \tag{2.2}$$

where g is called the *gauge coupling* and J is called *current*. The current is an operator that contains at least two fields. Our purpose in this section is to determine the possible form of the current J .

Requiring \mathcal{L}_{int} to be invariant under a gauge transformation implies

$$A_\mu J^\mu \rightarrow A_\mu J^\mu + \partial_\mu \xi(x) J^\mu$$

(by parts) (2.3)

$$A_\mu J^\mu - \xi(x) \partial_\mu J^\mu .$$

we see that \mathcal{L}_{int} is invariant under the gauge transformation (i.e. it is not affected by the unphysical longitudinal degree of freedom) only if

$$\partial_\mu J^\mu = 0 \quad \Leftrightarrow \quad \text{gauge invariance} \tag{2.4}$$

Stated in another way:

Massless vectors can only couple to conserved currents.

As a reminder, in the presence of a conserved current $\partial_\mu J^\mu = 0$ we have a conserved charge

$$Q = \int d^3x J^0 . \quad (2.5)$$

This is conserved in the sense that

$$\dot{Q} = \int d^3x \partial_0 J^0 = - \int d^3x \nabla \cdot \mathbf{J} = 0 , \quad (2.6)$$

where in the last step we have assumed the 3-current \mathbf{J} to vanish sufficiently quickly at infinity and used Gauss theorem. Conserved currents are a natural outcome of symmetries, as we are now going to explore in detail.

2.2 Symmetries

Since we are talking about fields, the central result is Noether theorem: to any continuous symmetry of the action corresponds a conserved current. We sketch below the proof of the result. Continuous symmetries are transformations that depend continuously on a parameter,

$$\phi(x) \rightarrow \phi_\lambda(x) . \quad (2.7)$$

Since the parameter is continuous, we can consider infinitesimal transformations for which $\lambda \ll 1$, and write

$$\phi(x) \rightarrow \phi(x) + \lambda \delta\phi(x) . \quad (2.8)$$

The crucial observation to prove Noether theorem is to observe that the action S is invariant if the variation of the Lagrangian under a symmetry transformation is a total derivative,

$$\delta\mathcal{L} = \lambda \partial_\mu F^\mu . \quad (2.9)$$

The variation of the Lagrangian is given by

$$\begin{aligned} \delta\mathcal{L} &= \mathcal{L}(\phi + \lambda\delta\phi, \partial\phi + \lambda\delta\partial\phi) - \mathcal{L}(\phi, \partial\phi) \\ &\simeq \lambda \left[\frac{\delta\mathcal{L}}{\delta\phi} \delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} \partial_\mu\delta\phi \right] . \end{aligned} \quad (2.10)$$

Using now the EoM $\delta\mathcal{L}/\delta\phi = \partial_\mu(\delta\mathcal{L}/\delta\partial_\mu\phi)$ we obtain

$$\begin{aligned} 0 &= \delta\mathcal{L} - \lambda\partial_\mu F^\mu \\ &= \left(\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} \right) \delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} \partial_\mu\delta\phi - \partial_\mu F^\mu \\ &= \partial_\mu \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} \delta\phi - F^\mu \right], \end{aligned} \quad (2.11)$$

which suggests the identification of the conserved current with

$$\boxed{J^\mu \equiv \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} \delta\phi - F^\mu.} \quad (2.12)$$

Notice that the expression of the current is not unique, since we can always shift $J^\mu \rightarrow J^\mu + \partial_\alpha A^{\alpha\mu}$, where $A^{\alpha\mu}$ is an antisymmetric object. Nevertheless, it is easy to show that the charge Q is not affected by the shift (assuming as usual that $A^{\alpha\mu}$ vanishes at infinity).

What happens at the quantum level? The field ϕ is now a quantum operator, and we know from Quantum Mechanics that the action of symmetries on operators is of the form

$$U^\dagger(\lambda)\phi U(\lambda) = \phi_\lambda, \quad (2.13)$$

where for continuous symmetries $U(\lambda)$ is a unitary matrix that can be written as

$$U(\lambda) = e^{-i\lambda Q}. \quad (2.14)$$

The hermitian operator Q is the *generator* of the transformation. At the infinitesimal level we have

$$\begin{aligned} (1 + i\lambda Q)\phi(1 - i\lambda Q) &\simeq \phi + \lambda\delta\phi \\ \phi + i\lambda [Q, \phi] &\simeq \phi + \lambda\delta\phi \end{aligned} \quad (2.15)$$

which implies

$$\boxed{\delta\phi \simeq i [Q, \phi].} \quad (2.16)$$

We say that Q *generates* the transformation. Remembering the Heisenberg EoM

$$\dot{Q} = i [H, Q] \quad (2.17)$$

we conclude that Q is conserved when it commutes with the Hamiltonian H . It is immediate to show that this is equivalent to requiring

$$U^\dagger(\lambda) H U(\lambda) = H, \quad (2.18)$$

i.e. Q is conserved only when the Hamiltonian is invariant under the transformation generated by Q .

The previous discussion applies to single particle QM. Does anything change in QFT? Since operators are local, we can always write

$$H = \int d^3x \mathcal{H}(x), \quad Q = \int d^3x \mathcal{Q}(x), \quad (2.19)$$

where \mathcal{H} is a usual the Hamiltonian density, while \mathcal{Q} is the *charge density*. We have immediately that the invariance of H under the symmetry implies

$$U^\dagger(\lambda)\mathcal{H}U(\lambda) = \mathcal{H} \Rightarrow [\mathcal{H}, Q] = 0. \quad (2.20)$$

What happens with the commutator between H and Q ? From charge conservation the most we can conclude is that

$$\dot{Q} = i[H, Q] = i \int d^3x [H, \mathcal{Q}(x)] = 0 \Rightarrow [H, \mathcal{Q}(x)] = i\nabla \cdot \mathbf{J}(x) \quad (2.21)$$

for some field $\mathbf{J}(x)$. The factor of i is inserted to make both sides anti-hermitian. Remembering that the left hand side is simply proportional to \dot{Q} apart from a factor of i , we conclude that even in QFT is true that

$$\boxed{\dot{Q} = -\nabla \cdot \mathbf{J}}, \quad (2.22)$$

which is the quantum version of Noether theorem.

The crucial question is now: can we compute the conserved quantum current using the classical formula coming from Noether theorem? The answer is yes. To convince ourselves that this is the case, we consider the explicit example of a Dirac fermion field.

2.3 Example: conserved current of a Dirac fermion

We start from the free Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi. \quad (2.23)$$

The Lagrangian is invariant under the phase transformation

$$\psi \rightarrow e^{-i\alpha}\psi, \quad (2.24)$$

which at the infinitesimal level amounts to

$$\psi \rightarrow \psi - i\alpha\psi. \quad (2.25)$$

We thus want to determine a charge operator such that

$$\delta\psi(x) = -i\psi(x). \quad (2.26)$$

Since the only operators available are ψ and ψ^\dagger (or $\bar{\psi}$), the charge operator will be a function of such operators:

$$Q = \int d^3x \mathcal{Q}(\psi_x, \psi_x^\dagger), \quad (2.27)$$

where, for simplicity of notation, we denote all the fields by $\mathcal{O}(x) = \mathcal{O}_x$. Let us try the simplest possibilities:

- $\mathcal{Q}_x = \psi_x$: using the anticommutation relations $\{\psi_x, \psi_y^\dagger\} = \delta_{xy}^3$ we cannot simplify the expression, which cannot thus be reduced to Eq. (2.26);


- $\mathcal{Q}_x = \psi_x^\dagger$: again the anticommutation relation does not help and we cannot obtain Eq. (2.26);
- $\mathcal{Q}_x = \psi_x^\dagger \psi_x$: in this case we obtain

$$\begin{aligned}
 \delta\psi_x &= i \int d^3y [\psi_y^\dagger \psi_y, \psi_x] \\
 &= i \int d^3y [\psi_y^\dagger \{\psi_y, \psi_x\} - \{\psi_y^\dagger, \psi_x\} \psi_y] \\
 &= -i \int d^3y \delta_{xy}^3 \psi_y \\
 &= -i\psi_x,
 \end{aligned} \tag{2.28}$$

which is exactly the result we wanted to obtain.

We can now use the Dirac equation to simplify the time derivative of $\mathcal{Q}_x = \psi_x^\dagger \psi_x = \bar{\psi}_x \gamma^0 \psi_x$, obtaining

$$\dot{\mathcal{Q}}_x = -\nabla \cdot (\bar{\psi}_x \boldsymbol{\gamma} \psi_x). \tag{2.29}$$

 **Exercise 2.1** Derive the previous result using the Dirac equation.

We thus deduce that the conserved 4-current associated to the transformation of Eq. (2.26) is

$$J^\mu = \bar{\psi} \gamma^\mu \psi. \tag{2.30}$$

Let us now compute the Noether current and show that we obtain the same result at the classical level. We start with the definition

$$J^\mu = \pi^\mu \delta\phi - F^\mu, \quad \text{with } \partial_\mu F^\mu = \delta\mathcal{L}. \tag{2.31}$$

We compute in turn the different terms:

- We start with $\delta\mathcal{L}$:

$$\begin{aligned}
 \delta\mathcal{L} &= \mathcal{L}' - \mathcal{L} \simeq (\bar{\psi} + i\alpha\bar{\psi}) (i\boldsymbol{\not{\partial}} - m) (\psi - i\alpha\psi) - \mathcal{L} \\
 &\simeq i\alpha (\bar{\psi} (i\boldsymbol{\not{\partial}} - m)\psi - \bar{\psi} (i\boldsymbol{\not{\partial}} - m)\psi) \\
 &\simeq 0,
 \end{aligned} \tag{2.32}$$

so that we can choose $F^\mu = 0$;

- The conjugate 4-momentum π^μ is given by

$$\pi^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi} = i\bar{\psi} \gamma^\mu. \tag{2.33}$$

We thus obtain that the Noether current is equal to

$$J^\mu = \bar{\psi} \gamma^\mu \psi, \tag{2.34}$$

exactly the result obtained in the quantum theory.

2.4 Antiparticles

After our discussion of the charge operator we are finally in the position to discuss antiparticles. Let us first remind that a real field is written as

$$\varphi_A(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_{\lambda} \left(u_A^{\lambda}(p) \alpha_{\lambda}(p) e^{-ipx} + u_A^{\lambda*}(p) \alpha_{\lambda}^{\dagger}(p) e^{ipx} \right), \quad (2.35)$$

while for complex fields we write

$$\varphi_A(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_{\lambda} \left(u_A^{\lambda}(p) \alpha_{\lambda}(p) e^{-ipx} + v_A^{\lambda}(p) \beta_{\lambda}^{\dagger}(p) e^{ipx} \right). \quad (2.36)$$

The difference between the two expressions lies in the second term, which for complex fields is not constrained to be the complex conjugate of the first one. We also remind the reader that the second term is the one associated with the negative-energy solutions. To keep our notation compact, we will write

$$\varphi(x) = \varphi^{(+)}(x) + [\varphi^{(+)}(x)]^{\dagger} \quad (2.37)$$

for a real field and

$$\varphi(x) = \varphi^{(+)}(x) + \varphi^{(-)}(x) \quad (2.38)$$

for a complex scalar field, leaving implicit all the Lorentz indices.

Suppose now we consider a particle which is charged under some symmetry. This means that its one particle states are eigenstates of some charge operator

$$e^{iQ} |\mathbf{p}\rangle = e^{iq} |\mathbf{p}\rangle, \quad (2.39)$$

where q is the charge of the particle (a number). At the level of creation and annihilation operators this can immediately be written as

$$[Q, \alpha^{\dagger}(p)] = +q \alpha^{\dagger}(p), \quad [Q, \alpha(p)] = -q \alpha(p). \quad (2.40)$$

This implies that the fields with positive and negative energy will transform as

$$\begin{aligned} [Q, \varphi^{(+)}(x)] &= -q \varphi^{(+)}(x), \\ [Q, \varphi^{(-)}(x)] &= +q \varphi^{(-)}(x). \end{aligned} \quad (2.41)$$

Let us now see what happens with the total field operator. We remind that our purpose is to write (Eq. (2.16))

$$[Q, \varphi(x)] = -i \delta\varphi(x), \quad (2.42)$$

where the right hand side must be expressed in terms of $\varphi(x)$ only. It is clear that in the case of a real field we have

$$\begin{aligned} [Q, \varphi(x)] &= [Q, \varphi^{(+)}(x)] + [Q, [\varphi^{(+)}(x)]^{\dagger}] \\ &= -q \varphi^{(+)}(x) + q [\varphi^{(+)}(x)]^{\dagger}. \end{aligned} \quad (2.43)$$

The only way to write this expression in terms of $\varphi(x)$ only is to admit $q = 0$, i.e. a real field cannot carry any non-trivial charge and cannot describe charged particles.

What happens in the case of a charged field? The situation now is different, since there is nothing forcing the negative-energy part $\varphi^{(-)}(x)$ to have the same charge. More specifically, we write

$$\begin{aligned} [Q, \varphi(x)] &= [Q, \varphi^{(+)}(x)] + [Q, \varphi^{(-)}(x)] \\ &= -q_{(+)} \varphi^{(+)}(x) + q_{(-)} \varphi^{(-)}(x) . \end{aligned} \quad (2.44)$$

We see that if we admit opposite charges,

$$q_{(-)} = -q_{(+)} \equiv -q \quad (2.45)$$

then the right hand side of the expression can be recombined and be written in terms of the total field $\varphi(x)$ as

$$[Q, \varphi(x)] = -q \varphi(x) . \quad (2.46)$$

We thus conclude that complex fields can carry charge, i.e. can create and destroy charged particles. The consequence of the last point is of paramount importance: every time we introduce an operator α^\dagger that creates a particle with a non-trivial charge, we must also introduce an operator β^\dagger that creates particles with *opposite* charge. Remembering that the $\varphi^{(-)}(x)$ field satisfies the same KG equation as the $\varphi^{(+)}(x)$ field, we conclude that this opposite charged particle created by β^\dagger has the same mass as the particle created by α^\dagger . This particle is called **antiparticle**.

To summarize, we have concluded that as a consequence of causality (i.e. of the fact that we need to write the field as in Eqs. (2.37) and (2.38)), the existence of particles carrying non-trivial charge implies the existence of antiparticles with same mass but opposite charge.

2.5 Scattering and Feynman rules

Once interactions are introduced, we open up the possibility of studying scattering and decays in QFT. Let us give some details of what happens. The two most important observables are the differential *cross section* associated to the process $p_1 + p_2 \rightarrow q_1 + q_2 + \dots$,

$$d\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \frac{1}{\mathcal{S}} |\mathcal{A}|^2 d\Phi_n , \quad (2.47)$$

and the decay width of an unstable particle $p \rightarrow q_1 + q_2 + \dots$,

$$d\Gamma = \frac{1}{2M} \frac{1}{\mathcal{S}} |\mathcal{A}|^2 d\Phi_n , \quad (2.48)$$

where M is the mass of the mother particle. $|\mathcal{A}|^2$ is the squared matrix element, that will depend on the theory under consideration. We will devote a good part of this section to see how to compute this quantity. In both the expressions above, $\mathcal{S} = N!$ where N is the number of identical particles in the final state, while $d\Phi_n$ is the n -particle phase space:

$$d\Phi_n = \prod_i \frac{d^3q_i}{(2\pi)^3 2E_i} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - \dots), \quad (2.49)$$

where the product is taken over all the particles in the final state. We will compute several cross sections and decay widths in the following chapters. For reference, however, we collect here the expression of the 2-body phase space computed in the CM frame of the final particles:

$$d\Phi_2^{CM} = \frac{d\Omega}{32\pi^2} \sqrt{\left(1 - \frac{(m - m')^2}{s}\right) \left(1 - \frac{(m + m')^2}{s}\right)}, \quad (2.50)$$

where m and m' denote the masses of the particles in the final state, s is the squared CM energy and $d\Omega$ is the element of solid angle between the two final state particles. Since these formulas are derived in any book about QFT we will not show how the expressions are obtained.

Before moving on to the computation of \mathcal{A} , let us introduce two more important quantities:

- the *luminosity* \mathcal{L} , defined as

$$\mathcal{L} = \frac{N_{tot}}{\sigma}, \quad (2.51)$$

where N_{tot} is the total number of events and σ is the total cross section. The luminosity \mathcal{L} is usually given by the experiments, and allows to compute the total number of events once σ is known;

- the *branching ratio* in a specific channel BR_i . To define the branching ratio, we first compute the total decay width $\Gamma_T = \sum_i \Gamma_i$, where Γ_i is the decay width in one specific channel i . We then define

$$BR_i = \frac{\Gamma_i}{\Gamma_T}. \quad (2.52)$$

The branching ratio represents the probability for the unstable particle to decay in the channel i .

We finally move to the computation of \mathcal{A} . To this end, consider either a scattering or a decay between initial state $|i\rangle$ and final state $|f\rangle$. We define the *S-matrix* as the operator such that the amplitude for the $|i\rangle \rightarrow |f\rangle$ process can be written as

$$\mathcal{M} = \langle f | S | i \rangle. \quad (2.53)$$

It is customary to write the S-matrix as $S = 1 + iT$. The matrix element \mathcal{A} appearing

in the expressions for the cross section and decay width is then defined as

$$\mathcal{A}(i \rightarrow f) = \langle f | T | i \rangle. \quad (2.54)$$

To be more precise, we explicitly factorize a delta function to take into account 4-momentum conservation, and we define

$$T_{fi} = (2\pi)^4 \delta(P_{in} - P_{fin}) \mathcal{A}_{i \rightarrow f}. \quad (2.55)$$

It can be shown that the S-matrix can be written as

$$S = T \exp \left(-i \int d^4x \mathcal{H}_{int}(x) \right) = 1 - i \int d^4x \mathcal{H}_{int}(x) + \frac{1}{2} \int d^4x \int d^4y T(\mathcal{H}_{int}(x) \mathcal{H}_{int}(y)) + \dots \quad (2.56)$$

The series is called *Dyson series*, and relies on the assumption that the interaction Hamiltonian \mathcal{H}_{int} depends on some small parameter over which we are doing perturbation theory. This is motivated by the fact that there are no known examples in 4-dimensions of interacting theories for which we can compute exactly the S-matrix. The interaction Hamiltonian contains terms which are of cubic or higher order in the quantum fields. Examples are $\mathcal{H}_{int}(x) = \lambda \phi(x)^4$ or $\mathcal{H}_{int}(x) = g A_\mu(x) \bar{\psi}(x) \gamma^\mu \psi(x)$. In these examples the Dyson series is an expansion in the (hopefully) small parameters λ or g .

The T symbol in the formula above denotes the *time ordering* operator

$$T(O(x)O(y)) = \theta(x^0 - y^0)O(x)O(y) + \theta(y^0 - x^0)O(y)O(x). \quad (2.57)$$

The last ingredient we need in order to be able to do actual computations is *Wick theorem*.

To state the theorem, we define the *normal ordering* of operators

$$: \phi_1 \phi_2 := \text{all creation operators on the left}, \quad (2.58)$$

and the Wick contraction

$$\overline{\varphi_1 \varphi_2} \equiv D_F(x_1 - x_2) = \langle 0 | \varphi_1 \varphi_2 | 0 \rangle, \quad (2.59)$$

where D_F is called Feynman propagator. We will study in Section 2.6 how to practically compute such propagators. The statement of Wick theorem is

$$T(\varphi_1 \dots \varphi_N) = : \varphi_1 \dots \varphi_N + \text{all possible contractions (also multiple ones)} : \quad (2.60)$$

Using Dyson series combined with Wick theorem we can compute any matrix elements needed, once the interactions are given. From the practical point of view, Feynman realized that the computation can be simplified using the so-called Feynman rules, that are simply mnemonic rules that encode the content of the Dyson series. To fix the notation, let us remind that a generic quantum field can be written as

$$\varphi_A(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \sum_\lambda \left[u_A^\lambda(\mathbf{p}) a_\lambda(\mathbf{p}) e^{-ipx} + v_A^\lambda(\mathbf{p}) b_\lambda^\dagger(\mathbf{p}) e^{ipx} \right], \quad (2.61)$$

where a destroys particles, b^\dagger creates antiparticles and u, v are ‘‘polarizations’’ (spinors)

for spin 1/2 particles, polarization vectors for spin 1 particles etc).

The Feynman rules can be summarized as follows:

- Factors involving the external particles: using the expression for quantum fields in Eq. (2.61) we see that a wave function factor $u_A^\lambda(\mathbf{p})$ is associated with the $a_\lambda(\mathbf{p})$ operator (i.e. with the destruction of a particle) while $v_A^\lambda(\mathbf{p})$ is associated with $b_\lambda^\dagger(\mathbf{p})$, i.e. with the creation of an antiparticle. Conversely, the expression for $\varphi_A^\dagger(x)$ shows that the creation of a particle must be associated with $u_A^{\lambda\dagger}(\mathbf{p})$ while the destruction of an antiparticle must be associated with $v_A^{\lambda\dagger}(\mathbf{p})$. In the case of fermions, the wave functions with \dagger must be replaced by wave functions with the Dirac-bar;
- The propagator in momentum space is computed according to the procedure outlined in Sec. 2.6;
- Feynman rules for interactions are computed according to

$$i \frac{\delta^n \mathcal{L}}{\delta \phi_1 \dots \delta \phi_n},$$

where ϕ_1, \dots, ϕ_n denote potentially different fields. This rule of thumb can be justified by the path integral approach to QFT and is a practical way to take into account possible permutation factors of identical legs in the vertex.

Let us give a concrete example: the Feynman rules associated with $\mathcal{H}_{int}(x) = -\mathcal{L}_{int}(x) = -gA_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x)$:



$$= ig\gamma^\mu. \quad (2.62)$$

We will see more examples in the following chapters.

Once the amplitude is computed we need to compute its squared modulus in order to have an expression for the observables. In this process we usually encounter the following ‘‘polarization sums’’:

$$\begin{aligned} \sum_\lambda u_\lambda(\mathbf{p})\bar{u}_\lambda(\mathbf{p}) &= \not{p} + m, \\ \sum_\lambda v_\lambda(\mathbf{p})\bar{v}_\lambda(\mathbf{p}) &= \not{p} - m, \\ \sum_\lambda \epsilon_\mu^\lambda(\mathbf{p})\epsilon_\nu^{\lambda*}(\mathbf{p}) &= \begin{cases} -g_{\mu\nu} & \text{massless} \\ -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} & \text{massive} \end{cases} \end{aligned} \quad (2.63)$$

With these sum polarizations it is possible to eliminate all the wave function dependence for a dependence on momenta, masses and metric tensors, allowing for a much simpler computation.

2.6 Propagators and gauge fixing

As we saw in the previous section, a fundamental role in the computation of the matrix elements is played by propagators, i.e. the two-point-functions $\langle 0|T\varphi^\dagger(x)\varphi(y)|0\rangle$ that measure the amplitude for a particle to be created in x and destroyed in y . As a practical rule, we remind that the propagators can be computed from the quadratic (or free) part of the Lagrangian. We first write it schematically as

$$\mathcal{L}_{\text{free}} = \begin{cases} \varphi^\dagger(x)\mathcal{O}(\partial, m)\varphi(x) & \text{(complex)} \\ \frac{1}{2}\varphi(x)\mathcal{O}(\partial, m)\varphi(x) & \text{(real)} \end{cases} \quad (2.64)$$

for complex and real fields, respectively. To obtain the Lagrangian written in the form above integration by parts can be used. We then write the expression in momentum space with the replacement $\partial_\mu \rightarrow -ip_\mu$. The propagator is then given by

$$\text{propagator} = i \mathcal{O}(-ip, m)^{-1}, \quad (2.65)$$

In the case of massive scalar, fermion and vector fields we obtain

$$\begin{aligned} \text{---} \xrightarrow{p} \text{---} &= \frac{i}{p^2 - m^2}, \\ \xrightarrow{p} &= \frac{i}{\not{p} - m} = \frac{i(\not{p} + m)}{p^2 - m^2}, \\ \text{~~~~} \xrightarrow{p} \text{~~~~} &= \frac{-i}{p^2 - m^2} \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{m^2} \right). \end{aligned} \quad (2.66)$$

The only non-trivial result is the last one, which we will now derive in detail. Let us start with the Lagrangian in momentum space. It reads

$$\mathcal{L}_{\text{free}}^{\text{mom}} = V_\mu^\dagger \left[-(p^2 - m^2) g^{\mu\nu} + p^\mu p^\nu \right] V_\nu. \quad (2.67)$$

To invert the tensor appearing in this expression we observe that for any momentum p we can define two projectors

$$P_T^{\mu\nu} \equiv g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}, \quad P_L^{\mu\nu} \equiv \frac{p^\mu p^\nu}{p^2}, \quad (2.68)$$

which project in the transverse and longitudinal direction with respect to p . It is easy to check that P_T and P_L have the usual properties of projectors. Given a tensor written as

$$A^{\mu\nu} = aP_T^{\mu\nu} + bP_L^{\mu\nu}, \quad (2.69)$$

its inverse is given by

$$(A^{\mu\nu})^{-1} = \frac{1}{a}P_T^{\mu\nu} + \frac{1}{b}P_L^{\mu\nu}. \quad (2.70)$$

It is just a matter of algebra to see that applying this procedure to the tensor $-(p^2 - m^2)g^{\mu\nu} + p^\mu p^\nu$ exactly reproduces the propagator written above.

An important point that emerges from the inspection of the propagators in Eq. (2.66) is that while the $m \rightarrow 0$ limit is smooth in the scalar and fermion case, there is singularity

in the vector case. This is again a manifestation of the difference between the number of degrees of freedom in a massive (3) and in a massless (2) vector. We see immediately that the reason behind the singularity is the fact that the tensor $-p^2 g^{\mu\nu} + p^\mu p^\nu$ to be inverted in the massless case is purely transverse, $-p^2 g^{\mu\nu} + p^\mu p^\nu = -p^2 P_T^{\mu\nu}$, and cannot thus be inverted. To allow for the computation of the propagator of a massless vector we thus need to add “by hand” a longitudinal term in the Lagrangian. This term is usually called “gauge fixing term” because a longitudinal term necessarily breaks explicitly¹ gauge invariance. A common choice is to consider the Lagrangian

$$\mathcal{L}_{\text{massless vector}} = -\frac{1}{4}(V_{\mu\nu})^2 - \frac{1}{2\xi}(\partial_\mu V^\mu)^2. \quad (2.71)$$

The operator in position space is

$$\mathcal{O}(\partial) = \square g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu. \quad (2.72)$$

This gives the propagator

$$\text{wavy line with } p \rightarrow = \frac{-i}{p^2} \left(g^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right). \quad (2.73)$$

Different values of ξ correspond to different gauge choices. For instance $\xi = 1$ is usually called Feynman gauge.

2.7 Example: the fermion-fermion to fermion-fermion scattering

Let us put together all the ingredients we have presented in the previous sections and write the expression for the matrix element of a process. We take the interaction $\mathcal{H}_{\text{int}}(x) = -\mathcal{L}(x) = -g A_\mu(x) \bar{\psi}(x) \gamma^\mu \psi(x)$ and consider the process $\psi(p_1) \psi(p_2) \rightarrow \psi(p_3) \psi(p_4)$. In order to draw all possible diagrams we first fix the external particle lines and then try to connect them in all possible ways using the interactions at our disposal. For the current case we have

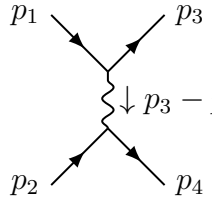


$$(2.74)$$

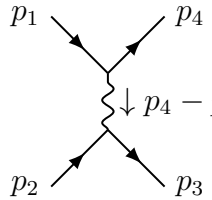
where the blob denotes all possible ways of connecting the fermion lines *using the vertex* and *respecting the directions of the arrows*. Since our vertex involves one outgoing fermion, one ingoing fermion and one massless vector, we see that we can connect the lines of the external particles in two ways: (i) p_1 with p_3 and p_2 with p_4 or (ii) p_1 with

¹We will see in the following chapters that there can also be a “spontaneous” breaking.

p_4 and p_2 with p_3 . We will thus have two independent amplitudes, each corresponding to one of the possible connections:



$$p_3 - p_1 = \mathcal{A}_1 = [\bar{u}_3(ig\gamma^\mu)u_1] D^{\mu\nu}(p_3 - p_1) [\bar{u}_4(ig\gamma^\nu)u_2]. \quad (2.75)$$



$$p_4 - p_1 = \mathcal{A}_2 = [\bar{u}_4(ig\gamma^\mu)u_1] D^{\mu\nu}(p_4 - p_1) [\bar{u}_3(ig\gamma^\nu)u_2]. \quad (2.76)$$

where $D^{\mu\nu}(p)$ is the massless vector propagator of E. (2.73). The overall amplitude for the process is given by the sum $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$.

Given a $2 \rightarrow 2$ process, it is customary to define the Mandelstam variables

$$\begin{aligned} s &= (p_1 + p_2)^2, \\ t &= (p_1 - p_3)^2, \\ u &= (p_1 - p_4)^2. \end{aligned} \quad (2.77)$$

Notice that only two of the Mandelstam variables are independent, since the following relation is always valid:

$$s + t + u = \sum_i m_i^2, \quad (2.78)$$

where the sum is taken over the masses of all the external particles. Going back to the amplitudes in Eqs. (2.75) and (2.76), we see that the squared photon momentum corresponds to the t variable in the first case and to the u variable in the second case. For this reason, the diagram in Eq. (2.75) is called *t-channel* while the one in Eq. (2.76) is called *u-channel*.

2.8 Effective Field Theories

Given a certain set of fields, we can write infinite interaction terms in \mathcal{L} . The question that arises is thus: **are all interactions equally relevant?**

To answer this question we need to keep in mind that

ALL THEORIES IN PHYSICS HAVE A LIMITED RANGE OF VALIDITY.

Stated in more technical terms, all theories in physics are Effective Theories (they are effective only in a certain range of validity). QFT is no exception, and every time we

write a QFT we are implicitly assuming that the theory we are considering is valid only for a limited range of energies/scales. In this case we use the term Effective Field Theory (EFT). This view (called Wilsonian) is a useful way to organize the terms in \mathcal{L} .

To be more quantitative, let us call $E_{max} \equiv \Lambda$ the maximum energy at which the EFT is valid. Λ is called *cutoff* of the EFT. Let us consider the theory of a massless spin-0 particle obeying a \mathbb{Z}_2 symmetry

$$\phi \rightarrow -\phi. \quad (2.79)$$

The Lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial\phi)^2 - \lambda_4\phi^4 - \lambda_6\phi^6 - \kappa_4\phi^2(\partial\phi)^2 - \kappa_6\phi^4(\partial\phi)^2 + \dots \\ &= \frac{1}{2}(\partial\phi)^2 - \lambda_4\phi^4 - \frac{g_6}{\Lambda^2}\phi^6 - \frac{h_6}{\Lambda^2}\phi^2(\partial\phi)^2 - \frac{h_8}{\Lambda^4}\phi^4(\partial\phi)^2 + \dots \end{aligned} \quad (2.80)$$

In the second line we have used dimensional analysis to write the dimensionful couplings in terms of the only scale available, the cutoff Λ .

Since dimensional analysis plays a fundamental role in our discussion, before turning to amplitudes it is important to discuss the dimension of an amplitude involving n particles. From Eqs. (2.55) and (2.53) we obtain immediately that

$$[\mathcal{A}_n] = 4 + [\mathcal{M}]. \quad (2.81)$$

To compute $[\mathcal{M}]$ we observe that an amplitude involving n particles involves n factors of (relativistically normalized) creation or annihilation operators. From Eq. (1.43) we immediately conclude that $[\mathcal{M}] = -n$, and finally

$$[\mathcal{A}_n] = 4 - n. \quad (2.82)$$

Let us now look at some amplitudes:

- 4-points amplitudes: according to the argument above, they must be dimensionless.

We thus write

$$\mathcal{A}_{2 \rightarrow 2} \sim \lambda_4 + \frac{h_6}{\Lambda^2} E^2 + \dots \quad (2.83)$$

where in the second term we have insert a typical energy scale for the process (for instance, it is of the order of the CM energy of the scattering) to make the dimensions right. We are not bothering in keeping track of the numerical factors appearing in the Feynman rules because they are not important for our argument. We see that at sufficiently low energies we have

$$\mathcal{A}_{2 \rightarrow 2} \rightarrow \lambda_4 \quad \text{as } E \rightarrow 0, \quad (2.84)$$

i.e. only the lowest dimensional coupling survives;

- 6-points amplitudes:

$$\mathcal{A}_{2 \rightarrow 4} \sim \frac{g_6}{\Lambda^2} + \left(\lambda_4 + \frac{h_6 E^2}{\Lambda^2} \right)^2 \frac{1}{E^2} \sim \frac{g_6}{\Lambda^2} + \frac{\lambda_4^2}{E^2} + \frac{h_6^2 E^2}{\Lambda^2} + 2 \frac{\lambda_4 h_6}{\Lambda^2}, \quad (2.85)$$

where in the first term we have inserted the (massless) scalar propagator necessary to obtain the 6-point amplitude. Again, we see that at low energies

$$\mathcal{A}_{2 \rightarrow 4} \rightarrow \frac{\lambda_4^2}{E^2} \quad \text{as } E \rightarrow 0, \quad (2.86)$$

i.e. only the lowest dimensional coupling survives.

This suggests to organize the Lagrangian as

$$\mathcal{L} = \underbrace{\mathcal{L}_{kin} + \mathcal{L}_4}_{\text{survives when } E \rightarrow 0} + \frac{1}{\Lambda} \mathcal{L}_5 + \frac{1}{\Lambda^2} \mathcal{L}_6 + \dots \quad (2.87)$$

The part of the Lagrangian that survives as $E \rightarrow 0$ is called “renormalizable” for historical reasons. This means that we can organize our discussion in inverse powers of Λ , and start our discussion with the renormalizable Lagrangian.

The fact that only a limited number of terms are important in the low energy/large distance limit is not peculiar of QFT. Another familiar example is given by the potential of a static distribution of charge of typical size a . We know from classical electrodynamics that at a distance $R \ll a$ we have

$$\Phi = \frac{Q_0}{R} + \frac{Q_1^i R^i}{R^3} + \frac{Q_2^{ij} R^i R^j}{R^5} + \dots \quad (2.88)$$

where the terms refer to the monopole, dipole, quadrupole and higher order multipoles, respectively. Each multipole grows with the size of the object according to $Q_\alpha \sim a^\alpha$. It is clear that a measurement performed at very large distances will be sensitive only to the monopole term (i.e. from very far away, every charge distribution looks like a point-like charge). Diminishing the distance R between the charge and the experimental apparatus allows us to probe higher multipoles: first the dipole, then the quadrupole etc. Clearly, when $R \sim a$ the expansion completely breaks down, and we need the full expression of the potential to make predictions (i.e. we need to change our description). Exactly the same reasoning is behind the idea of EFT in QFT: for energies $E \ll \Lambda$ only the operators of smallest dimensions will be probed by experiments. As the energy increases, more and more operators are necessary, until we reach $E \sim \Lambda$ and a new description is needed.

The analogy with the electrostatic potential highlight also another fundamental fact about the long distance/high energy limit: since fewer terms are relevant, accidental symmetries appear which are not symmetries of the whole problem. In Eq. (2.88) the monopole term is completely invariant under rotations, since it depends only on the magnitude of the distance R . The symmetry of the monopole is thus $SO(3)$. When the dipole is important, we see we do not have anymore the freedom of performing

arbitrary rotations: only those that will leave invariant Q_1^i will be symmetries of the system. The symmetry is hence reduced to $SO(2)$, the group of rotations around a fixed axis (determined by Q_1^i). Finally, when the quadrupole term is important, also the $SO(2)$ symmetry is lost. We will be back on the point of symmetries many times in the following chapters.


2.9 Additional exercises

 **Exercise 2.2** Consider the following interaction Lagrangian:

$$\mathcal{L}_{int} = y\phi\bar{\psi}\psi + h.c. \quad (2.89)$$

where ϕ is a real scalar and ψ a Dirac Fermion. They have masses m_ϕ and m_ψ , respectively.

- Compute the cross section $\psi\bar{\psi} \rightarrow \psi\bar{\psi}$;
- Compute the decay width $\phi \rightarrow \psi\bar{\psi}$.

 **Exercise 2.3** Repeat the computations of the previous exercise considering the following Lagrangian between a massive vector V_μ and a Dirac fermion ψ :

$$\mathcal{L}_{int} = yV_\mu\bar{\psi}\gamma^\mu\psi + h.c. \quad (2.90)$$

2.10 Additional reading

- S.Weinberg, “The Quantum Theory of Fields” vol. 1;
- M.Peskin and D.Schroeder, “An Introduction To Quantum Field Theory”;
- S.Coleman, “Lectures on Quantum Field Theory”;
- M.Schwartz, “Quantum Field Theory and the Standard Model”;
- D. B. Kaplan, “Lectures on Effective Field Theory” ([link](#));
- R. Penco, “An Introduction to Effective Field Theories” ([link](#));
- T.Cohen, “As scales become separated: lectures on effective field theory” ([link](#)).

Chapter 3 Quantum Electrodynamics

In nature, Quantum Electrodynamics is the theory of a massive electron (or, more in general, of massive charged particles) and of a massless photon. To construct the Lagrangian of the theory we will start from the free particle Lagrangians of the electron and of the photon, and add a linear coupling between A_μ and a conserved current made out of electrons. As we know from the previous chapter, the free fermion Lagrangian is invariant under a phase transformation, and admits a Noether current. Overall we obtain

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \bar{\psi} (i\cancel{D} - m) \psi - eqA_\mu \bar{\psi} \gamma^\mu \psi, \quad (3.1)$$

where e is the electric charge (defined positive) and $q = -1$ is the actual electron charge. This notation is convenient because it is customary to write all electric charged in units of the electron charge (i.e. factorizing explicitly a factor of e).

3.1 Covariant derivatives

A more compact way to write the Lagrangian of Eq. (3.1) is by means of the idea of “covariant derivative”

$$D_\mu \psi \equiv \partial_\mu \psi + ieqA_\mu \psi. \quad (3.2)$$

In terms of the covariant derivative the Lagrangian simply reads

$$\mathcal{L} = \bar{\psi} (i\cancel{D} - m) \psi. \quad (3.3)$$

In our approach, the covariant derivative is simply a compact way of writing the Lagrangian. There is however another way to get to the same result, by invoking the so-called “gauge principle”:

1. promote the phase transformation under which the free fermion Lagrangian is invariant to a *local* transformation

$$\psi(x) \rightarrow e^{i\alpha(x)} \psi(x); \quad (3.4)$$

2. The free-fermion Lagrangian is no longer invariant under the local transformation, but it transforms according to

$$\bar{\psi}(i\cancel{D} - m)\psi \rightarrow \bar{\psi}(i\cancel{D} - m)\psi - \partial_\mu \alpha (\bar{\psi} \gamma^\mu \psi); \quad (3.5)$$

3. The easiest way to restore the invariance is to add the linear $-eq A_\mu \bar{\psi} \gamma^\mu \psi$ inter-

action. The Lagrangian is thus invariant under the *gauge transformation*

$$\begin{aligned}\psi(x) &\rightarrow e^{i\alpha(x)}\psi(x) \\ A_\mu(x) &\rightarrow A_\mu(x) - \frac{1}{eq}\partial_\mu\alpha.\end{aligned}\tag{3.6}$$

In this simple case the two approaches (adding a linear coupling between the photon and a conserved current and requiring invariance under a local phase transformation) are completely equivalent. We will see soon that in more complicated cases like scalar QED the gauge principle allows to get to the correct answer in a quicker a simpler fashion. It is worth however to stress that the gauge principle is nothing more than a recipe to ensure that the physics is invariant under the gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu\xi$.

Notice that it is inconvenient to have the photon transformation depending on the fermion charge, while the fermion transformation does not depend on q . This may be problematic if more charged particles are present, since the photon transformation cannot make the Lagrangian invariant. The simple way out is to redefine $\alpha(x) \rightarrow q\alpha(x)$. We thus obtain that the gauge transformation is

$$\begin{aligned}\psi(x) &\rightarrow e^{iq\alpha(x)}\psi(x) \\ A_\mu(x) &\rightarrow A_\mu(x) - \frac{1}{e}\partial_\mu\alpha.\end{aligned}\tag{3.7}$$

Why is D_μ called a *covariant* derivative? To understand this point we need to look at how $D_\mu\psi$ transforms under the gauge transformation of Eq. (3.6):

$$\begin{aligned}D_\mu\psi &\rightarrow (\partial_\mu + ieqA_\mu - iq\partial_\mu\alpha)e^{iq\alpha(x)}\psi \\ &= (iq\partial_\mu\alpha)e^{iq\alpha}\psi + e^{iq\alpha}\partial_\mu\psi + ieqA_\mu e^{iq\alpha}\psi - (iq\partial_\mu\alpha)e^{iq\alpha}\psi \\ &= e^{iq\alpha(x)}D_\mu\psi,\end{aligned}\tag{3.8}$$

i.e. the covariant derivative inherits the same transformation as the object to which it is applied to. ¹

3.2 Amplitudes and gauge invariance

Another aspect of gauge invariance becomes clear if we look at the amplitudes for processes with photons in the external legs. In this case we need to make sure that we do not excite longitudinal degrees of freedom which are unphysical. We remind that longitudinal degrees of freedom are those whose polarization vectors are parallel to the momentum. Invariance under a gauge transformation has precisely the role to eliminate

¹A similar line of reasoning leads to covariant derivatives in general relativity.

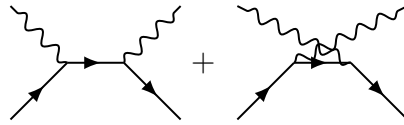
such degrees of freedom via the identification

$$\epsilon_\mu(p) \sim \epsilon_\mu(p) + \xi p_\mu . \quad (3.9)$$

When we consider processes with external photons the polarization vectors ϵ_μ will appear in the amplitude, we can check whether the amplitude vanishes once we substitute $\epsilon_\mu(p) \rightarrow p_\mu$. If the amplitude does not vanish, then the theory is not consistent, since longitudinal photons are interacting with other particles in the theory.

Let us consider Compton scattering as an explicit example, i.e. the process $e\gamma \rightarrow e\gamma$.

There are two diagrams contributing to the amplitude



$$(3.10)$$

The amplitude results (the momenta are fixed according to $e(p_1) + \gamma(q_1) \rightarrow e(p_2) + \gamma(q_2)$)

$$\mathcal{A} = \mathcal{M}^{\mu\nu} \epsilon_\mu^*(q_2) \epsilon_\nu(q_1) \quad (3.11)$$

with

$$\begin{aligned} \mathcal{M}^{\mu\nu} = & \bar{u}_{p_2} (ieq\gamma^\mu) i \frac{\not{p}_1 + \not{q}_1 + m}{(p_1 + q_1)^2 - m^2} (ieq\gamma^\nu) u_{p_1} \\ & + \bar{u}_{p_2} (ieq\gamma^\nu) i \frac{\not{p}_1 - \not{q}_2 + m}{(p_1 - q_2)^2 - m^2} (ieq\gamma^\mu) u_{p_1} . \end{aligned} \quad (3.12)$$

Gauge invariance amounts to

$$q_1^\nu \mathcal{M}^{\mu\nu} = 0 = q_2^\mu \mathcal{M}^{\mu\nu} . \quad (3.13)$$

Let's check that this identities are true. We first observe that the denominators are equal to

$$\begin{aligned} (p_1 + q_1)^2 - m^2 &= m^2 + 2p_1 \cdot q_1 - m^2 = 2p_1 \cdot q_1 \\ (p_1 - q_2)^2 - m^2 &= m^2 - 2p_1 \cdot q_2 - m^2 = -2p_1 \cdot q_2 \end{aligned} \quad (3.14)$$

To simplify the amplitude we will use the fact that the spinors satisfy the equations of motion

$$(\not{p} - m)u_p = 0 , \quad \bar{u}_p(\not{p} - m) = 0 \quad (3.15)$$

and the identity

$$\not{a}\not{b} = 2ab - \not{b}\not{a} . \quad (3.16)$$

We thus have

$$\begin{aligned}
 q_2^\mu \mathcal{M}^{\mu\nu} &= \bar{u}_{p_2} \left[\frac{\not{q}_2(\not{p}_2 + \not{q}_2 + m)\gamma^\nu}{2p_1q_1} - \frac{\gamma^\nu(\not{p}_1 - \not{q}_2 + m)\not{q}_2}{2p_1q_2} \right] u_{p_1} \\
 &= \bar{u}_{p_2} \left[\frac{\not{q}_2\not{p}_2 + m\not{q}_2}{2p_2q_2} - \gamma^\nu \frac{\not{p}_1\not{q}_2 + m\not{q}_2}{p_1q_2} \right] u_{p_1} \\
 &= \bar{u}_{p_2} \left[\frac{2p_2q_2 - (\not{p}_1 - m)\not{q}_2}{2p_2q_2} \gamma^\nu - \gamma^\nu \frac{2p_1q_2 - \not{q}_2(\not{p}_1 - m)}{2p_1q_2} \right] u_{p_1} \\
 &= 0
 \end{aligned} \tag{3.17}$$

using the EoM. The same procedure applies when we contract with q_1^ν .

3.3 Scalar QED

We will now discuss how to write the theory of a spin-0 massive particle interacting with a massless photon. We will use the amplitude approach of the previous section. As usual, we will start by coupling linearly the photon with the conserved current associated with

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2, \tag{3.18}$$

which reads

$$J^\mu = i (\partial^\mu \phi^\dagger \phi - \phi^\dagger \partial^\mu \phi). \tag{3.19}$$

Our starting point is thus the Lagrangian

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2 - \frac{1}{4} (F_{\mu\nu})^2 + eq A_\mu i (\partial^\mu \phi^\dagger \phi - \phi^\dagger \partial^\mu \phi). \tag{3.20}$$

The amplitude for the process under consideration is



$$\tag{3.21}$$

It is clear that we need the Feynman rule for the scalar-scalar-photon vertex. Feynman rules for derivative interactions are not completely straightforward to derive. We will use a simple trick, that will give us the correct answer. Imagine each scalar line to be associated to a particle (and not to an antiparticle). We schematically write

$$\phi \sim a e^{-ipx}, \quad \phi^\dagger \sim a^\dagger e^{ipx}, \tag{3.22}$$

where we are not writing the integration over the momenta and the various numerical factors appearing in the complete expression. Derivatives acting on such fields give

$$\partial_\mu \phi \sim -ip_\mu \phi, \quad \partial_\mu \phi^\dagger \sim ip_\mu \phi^\dagger, \tag{3.23}$$

i.e. derivatives produce factors of momenta. In particular, a factor ip is associated with the creation of a particle of momentum p , while $-ip$ is associated with the destruction

of a particle with momentum p . The Feynman rule can thus be computed considering the usual factor of i from Dyson formula and a factor of ieq present in the interaction Lagrangian, becoming

$$\begin{array}{c}
 \text{wavy line} \\
 \updownarrow \\
 p_{in} \rightarrow \text{---} p_{out} \rightarrow
 \end{array}
 = i(ieq) (ip_{out}^\mu + ip_{in}^\mu) = -ieq(p_{in} + p_{out})^\mu . \quad (3.24)$$

We are now in the position of computing the amplitude for Compton scattering. We obtain

$$\mathcal{A} = \mathcal{M}_{scalar}^{\mu\nu} \epsilon_\mu^*(q_2) \epsilon_\nu(q_1) , \quad (3.25)$$

with

$$\mathcal{M}_{scalar}^{\mu\nu} = -i(-ieq)^2 \left[\frac{(2p_2 + q_2)^\nu (2p_1 + q_1)^\mu}{(p_1 + q_1)^2 - m^2} + \frac{(2p_2 - q_1)^\mu (2p_1 - q_2)^\nu}{(p_1 - q_2)^2 - m^2} \right] . \quad (3.26)$$

Let us now check gauge invariance. Contracting with q_2^ν we obtain

$$\begin{aligned}
 q_2^\nu \mathcal{M}_{scalar}^{\mu\nu} &= ie^2 q^2 q_2^\nu \left[\frac{(2p_2 + q_2)^\nu (2p_1 + q_1)^\mu}{(p_1 + q_1)^2 - m^2} + \frac{(2p_2 - q_1)^\mu (2p_1 - q_2)^\nu}{(p_1 - q_2)^2 - m^2} \right] \\
 &= ie^2 q^2 \left[\frac{2p_2 q_2 (2p_1 + q_1)^\mu}{2p_1 q_2} - \frac{(2p_2 - q_1)^\mu 2p_1 q_2}{2p_1 q_2} \right] \\
 &= ie^2 q^2 (2p_1 - 2p_2 + 2q_1)^\mu \\
 &= 2ie^2 q^2 q_2^\mu .
 \end{aligned} \quad (3.27)$$

If we instead contract with q_1^μ we obtain

$$q_1^\mu \mathcal{M}_{scalar}^{\mu\nu} = 2ie^2 q^2 q_1^\nu . \quad (3.28)$$

As we can see, these expressions do not vanish, i.e. **gauge invariance is not respected in the Lagrangian of Eq. (3.20)**. What goes wrong? Inspecting Eq. (3.20) we see that we are adding derivative interactions to the Lagrangian, which modify the expression of the Noether current. This means that the Noether current we are coupling the photon to is not the correct one. To understand how to correct for this problem, we proceed systematically and try to first infer which term we need to add to $\mathcal{M}_{scalar}^{\mu\nu}$ to recover gauge invariance, and then we will translate them into new operators in the Lagrangian. We thus seek to write

$$\mathcal{M}_{tot}^{\mu\nu} = \mathcal{M}_{scalar}^{\mu\nu} + \mathcal{M}_{new}^{\mu\nu} , \quad (3.29)$$

with $\mathcal{M}_{new}^{\mu\nu}$ such to recover gauge invariance:

$$q_2^\nu \mathcal{M}_{tot}^{\mu\nu} = 0 , \quad q_1^\mu \mathcal{M}_{tot}^{\mu\nu} = 0 . \quad (3.30)$$

This translates into the conditions

$$\begin{aligned} q_2^\nu \mathcal{M}_{new}^{\mu\nu} &= -2ie^2 q^2 q_2^\mu, \\ q_1^\mu \mathcal{M}_{new}^{\mu\nu} &= -2ie^2 q^2 q_1^\nu. \end{aligned} \quad (3.31)$$

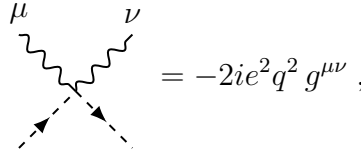
To infer the form of $\mathcal{M}_{new}^{\mu\nu}$ we proceed as follows: we first write all possible tensor structures that can be constructed out of q_1^μ and q_2^μ imposing gauge invariance. We obtain

$$\mathcal{M}_{new}^{\mu\nu} = \alpha g^{\mu\nu} + \beta q_1^\mu q_2^\nu. \quad (3.32)$$

We then observe that the dimension of the new constants appearing are $[\alpha] = 0$ and $[\beta] = -2$. Using our EFT arguments we know that in the very low energy limit only the term proportional to α will be relevant, meaning that it is this term alone that should be able to restore gauge invariance. This is true if

$$\mathcal{M}_{new}^{\mu\nu} = -2ie^2 q^2 g^{\mu\nu}. \quad (3.33)$$

This is giving us a very important information: to restore gauge invariance we must have a new vertex in the theory,



$$= -2ie^2 q^2 g^{\mu\nu}, \quad (3.34)$$

which must come from an addition term in the Lagrangian of the form

$$\mathcal{L}_{new} = -e^2 q^2 A_\mu A^\mu \phi^\dagger \phi. \quad (3.35)$$

The total Lagrangian thus results in

$$\begin{aligned} \mathcal{L} &= |\partial_\mu \phi|^2 - m^2 |\phi|^2 - \frac{1}{4} (F_{\mu\nu})^2 \\ &\quad + ieq A_\mu (\partial^\mu \phi^\dagger \phi - \phi^\dagger \partial^\mu \phi) - e^2 q^2 A_\mu A^\mu \phi^\dagger \phi \\ &= |\partial_\mu \phi + ieq A_\mu \phi|^2 - \frac{1}{4} (F_{\mu\nu})^2. \end{aligned} \quad (3.36)$$

We see that we can again write the Lagrangian in terms of the covariant derivative, as it already happened for spinor QED. This is a general result: the gauge principle provides us with a simple and systematic way to construct theories in which gauge invariance is guaranteed from the very beginning.

3.4 When a symmetry is not a symmetry: anomalies

As we saw at the beginning of this chapter, massless vectors must couple to conserved currents to guarantee that gauge invariance is respected. We derived the conserved current in the cases of spinor and scalar QED using Noether theorem, i.e. with a com-

pletely classical procedure. The natural question is now: *can quantum effects spoil current conservation?* Quite surprisingly, the answer is *yes*. When the conservation of a current is spoiled by quantum effects we say that the current is *anomalous*, or that the theory has an *anomaly*.

Anomalies are a very deep and quite complicated subject of QFT, so we will not enter into the details of any computation, simply limiting ourselves to give a few “recipes” useful to establish whether a current is anomalous or not. More informations can be found in any advanced QFT book. Another useful reference are [these](#) lecture notes.

Consider a situation in which we have a current J_A^μ associated with a charge Q_A , and we want to understand whether this current is anomalous. In QED we then obtain

$$\partial_\mu J_A^\mu = \left[\sum_{f_L} Q_A^{f_L} (Q_e^{f_L})^2 - \sum_{f_R} Q_A^{f_R} (Q_e^{f_R})^2 \right] \frac{e^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} . \quad (3.37)$$

In the previous expression the sum over f_L is over all left handed fermions, while the sum over f_R is over all right handed fermions, with Q_A and Q_e , respectively, their charges associated with the J_A and electromagnetic current. The result is valid both for “global” current (i.e. currents associated to symmetries of the Lagrangian not coupled to any massless vector) and for “local” currents (i.e. currents coupled to some massless vector).

We see that as long as any fermion has a left handed and right handed component with equal charges the anomaly vanishes. This is the case in spinor QED², and every time this happens we say that the fermion is “vector-like”. More general situations will emerge later on in our study of electroweak interactions. When the left handed and right handed components transform in different ways the fermions are called chiral and so is the theory.³

Another type of anomaly emerges from the coupling with gravity (gravitational anomalies). In a quantum theory the graviton is the massless spin 2 mediator of the gravitational force, and it is usually embedded in a symmetric tensor $h_{\mu\nu}$ for a Lorentz invariant description. Since $h_{\mu\nu}$ has 10 independent components, while the physical graviton has only two polarizations as the photon, it is clear that a gauge redundancy must be present to take care of the redundant degrees of freedom. The condition that ensures that there are no gravitational anomalies is

$$\sum_{f_L} Q_A^{f_L} - \sum_{f_R} Q_A^{f_R} = 0 . \quad (3.38)$$

This condition must always be satisfied for the theory to have the possibility of being

²Notice that there is nothing to be checked in scalar QED, since it is a theory without fermions.

³It is important to observe that, from the point of view of Lorentz symmetry, left and right handed fermions are two different objects. This allows to give them different quantum numbers without spoiling Lorentz invariance.

consistently coupled to gravity.

3.5 Loops and renormalization

Up to this point we have always considered processes at *tree-level*. Loop processes are however of fundamental importance, and contain an extremely rich physics. As we are going to see, the Standard Model of particle physics is compatible with data only when loop processes are considered, making it one of the greatest successes of QFT. Loops are however a quite delicate subject, since they have the counterintuitive property of giving infinite contributions to amplitudes. For this reason, the process of making sense of the theory proceeds in two steps: (i) regularization (in which a well defined procedure is established to deal with infinities) and (ii) renormalization (in which the infinities are eliminated from the observables).

Once more we will not enter into too many details, but we will just point out a few essential facts:

- We will always use dimensional regularization, i.e. we will perform the loop integrals in $d = 4 - \epsilon$ dimensions. All the results will depend on ϵ , and we will take the $\epsilon \rightarrow 0$ limit only at the end of the computation, after having renormalized the theory;
- The parameters in the Lagrangian are not physical parameters but are called “bare” parameters. To connect them to physical quantities we need a renormalization procedure that connects them to some observable. This can be done in two ways:
 - counter terms: we write the bare parameters as $a_0 = a_R + \delta a$, where a_R is the (finite) renormalized parameter and δa is the (infinite) counter-term that cancel the loop divergencies;
 - direct renormalization: we trade the bare parameters for observables from the very beginning, without splitting them in renormalized and counter-terms.

In both cases the final observable will be finite and well defined. Both procedures can be found in the literature, and it is useful to be aware of the existence of both. It is probably useful to analyze an example to highlight the problem and the solution. We take QED as example, although what we are saying is generic. The loop corrections are of the form shown in Fig. 3.1. Let us focus on the photon propagator corrections, which will lead us to the idea of *running coupling*. We begin by denoting by e_0 the electric charge appearing in the QED Lagrangian in Eq. (3.1) to remember that it is a bare parameter. In addition, we redefine the photon field to absorb the electric charge,

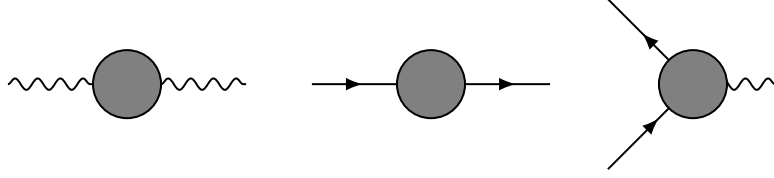


Figure 3.1: Radiative corrections in QED.

$A_\mu \rightarrow A_\mu/e_0$. We obtain

$$\mathcal{L} = \bar{\psi} (i\cancel{\partial} - m) \psi + A_\mu \bar{\psi} \gamma^\mu \psi - \frac{1}{4e_0^2} (F_{\mu\nu})^2 . \quad (3.39)$$

Written in this form, it is clear that the electric charge is connected to the photon propagator. As a matter of fact, it can be shown the vertex correction (right panel in Fig.3.1) and the electron propagator corrections (middle panel in Fig.3.1) cancel each other in the computation of any physical amplitude. To fix the notation, let us denote the blob appearing in the photon propagator by

$$\mu \xrightarrow{q} \text{blob} \xrightarrow{q} \nu = i\Pi^{\mu\nu}(q^2) = i(q^2\Pi(q^2)g^{\mu\nu} + \Pi'(q^2)q^\mu q^\nu) , \quad (3.40)$$

where q is the momentum flowing into the blob. The tensor decomposition follows remembering that the blob must be symmetric under the $\mu \leftrightarrow \nu$ exchange. Moreover, we have inserted a factor q^2 in the first term to make the dimensions of Π and Π' equal. The blob will contain all the loop corrections which are *one particle irreducible*⁴, at any given order (one loop, two loops etc). The blob in (3.40) will be contracted with photons, and must satisfy the usual gauge invariance conditions. In this case they read

$$q_\mu \Pi^{\mu\nu}(q^2) = 0 = q_\nu \Pi^{\mu\nu}(q^2). \quad (3.41)$$

We obtain

$$\begin{aligned} q_\mu \Pi^{\mu\nu}(q^2) &= q_\mu (q^2\Pi(q^2)g^{\mu\nu} + \Pi'(q^2)q^\mu q^\nu) \\ &= q^2 (\Pi(q^2) + \Pi'(q^2)) q^\nu \\ &\Rightarrow \Pi'(q^2) = -\Pi(q^2) \\ &\Rightarrow \Pi^{\mu\nu}(q^2) = q^2\Pi(q^2)P_T^{\mu\nu} , \end{aligned} \quad (3.42)$$

where P_T is the projector defined in Eq. (2.68). Notice that we did not take $q^2 = 0$ to allow for off-shell photons (as they are in a propagator). Before analyzing how the loop corrections affect the electric charge let us notice that the photon propagator following from Eq. (3.39) is

$$\text{wavy line } \xrightarrow{p} = \frac{-ie_0^2}{p^2} \left(g^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right) \equiv D_0^{\mu\nu} . \quad (3.43)$$

⁴One particle irreducible means that they cannot be separated into two disjointed diagrams by cutting any one of the internal lines.

We are now in a position to include all photon propagator corrections. To this end the photon propagator including radiative corrections is given by

$$\text{~~~~~} + \text{~~~~~} \text{---} \text{---} \text{---} + \text{~~~~~} \text{---} \text{---} \text{---} \text{---} \text{---} + \dots \quad (3.44)$$

Denoting the full propagator by $D^{\mu\nu}$ we have

$$\begin{aligned} D^{\mu\nu} &= D_0^{\mu\nu} + D_0^{\mu\alpha} \Pi^{\alpha\beta}(q^2) D_0^{\beta\nu} + \dots \\ &= -\frac{i e_0^2}{q^2} (P_T^{\mu\nu} + \xi P_L^{\mu\nu}) - i \frac{e_0^2}{q^2} [e_0^2 \Pi(q^2)]^2 P_T^{\mu\nu} - i \frac{e_0^2}{q^2} [e_0^2 \Pi(q^2)]^2 P_T^{\mu\nu} + \dots \\ &= -i \frac{e_0^2}{q^2} \left[1 + e_0^2 \Pi(q^2) + (e_0^2 \Pi(q^2))^2 + \dots \right] P_T^{\mu\nu} - i \frac{e_0^2}{q^2} \xi P_L^{\mu\nu} \\ &= -i \frac{e_0^2}{q^2} \frac{1}{1 - e_0^2 \Pi(q^2)} P_T^{\mu\nu} - i \frac{e_0^2}{q^2} \xi P_L^{\mu\nu}. \end{aligned} \quad (3.45)$$

We observe two interesting facts: (i) the gauge dependent part does not receive radiative corrections (because $P_T P_L = 0$) and (ii) the transverse part prompts us to define an energy-dependent coupling

$$e^2(q^2) \equiv \frac{e_0^2}{1 - e_0^2 \Pi(q^2)}. \quad (3.46)$$

This is the so-called *running coupling*, since it changes non-trivially with energy. We can now proceed with our renormalization program: we connect the running coupling to an observable and trade e_0 for such observable. Suppose the running coupling is measured at very small energies, as experimentally happens in real life ⁵. Inverting the previous equation, we can write

$$e_0^2 = \frac{e^2(0)}{1 + e^2(0)\Pi(0)}, \quad \Rightarrow \quad e^2(q^2) = \frac{e^2(0)}{1 - e^2(0)[\Pi(q^2) - \Pi(0)]}. \quad (3.47)$$

Any divergence present in $\Pi(q^2)$ will cancel out from the combination $\Pi(q^2) - \Pi(0)$, leaving the result finite and expressed solely in terms of observables.

Before concluding this section, let us compute explicitly $\Pi(q^2)$ in spinor QED at one loop. We can use the results of Appendix E, obtaining

$$\begin{aligned} \text{~~~~~} \text{---} \text{---} \text{---} &= i\Pi^{\mu\nu}(q^2) = iq^2 \Pi(q^2) P_T^{\mu\nu} \\ \Rightarrow \Pi(q^2) &= -\frac{Q_f^2}{36\pi^2} \left[5 + 3 \left(\frac{1}{\tilde{\epsilon}} + \log \frac{\mu^2}{m_f^2} \right) + \frac{12m_f^2}{q^2} + 3(2m_f^2 + q^2) f(q^2, m_f, m_f) \right] \end{aligned} \quad (3.48)$$

where Q_f is the fermion electric charge, $1/\tilde{\epsilon} = 1/\epsilon + \log(4\pi) - \gamma_E$ (with γ_E the Euler

⁵We will get back on this point in the next section

constant) and

$$f(q^2, m_f, m_f) = \frac{\sqrt{q^2(q^2 - 4m_f^2)}}{q^4} \log \left(\frac{2m_f^2 - q^2 + \sqrt{q^2(q^2 - 4m_f^2)}}{2m_f^2} \right). \quad (3.49)$$

 **Exercise 3.1** Derive the previous result using Package X.

When more than one charged fermion is considered we must sum over their individual contributions. As we can see from the previous result, the explicit loop computation is precisely a transverse tensor as expected from gauge invariance. Moreover, notice that the limit $q^2 \rightarrow 0$ is smooth, and we have

$$\Pi(q^2 = 0) = -\frac{Q_f^2}{12\pi^2} \left(\frac{1}{\epsilon} + \log \frac{\mu^2}{m_f^2} \right). \quad (3.50)$$

As expected, the combination $\Pi(q^2) - \Pi(0)$ is finite.

Let us notice that, according to the meaning of q^2 , different phenomena happen:

- suppose $q^2 = t$ or u , i.e. the photon propagator is in the t (or u)-channel of a process, and q^2 represent the exchange momentum between the initial and final states. It is immediate to check that in this case $q^2 < 0$, and we write $q^2 = -Q^2$ with $Q^2 > 0$. Inspecting Eq. (3.48) we see that in this regime the argument of the square root and of the logarithm are always positive, and $e^2(Q^2)$ is an increasing function of Q^2 . The electric charge hence *increases as the energy increases or, equivalently, as the distance diminishes*;
- suppose now $q^2 = s$, i.e. the photon propagator is in the s-channel of a process. In this case $q^2 > 0$, and we see that the argument of the logarithm becomes negative for $q^2 > 4m_f^2$, signal the existence of a branching cut beginning at $q^2 = 4m_f^2$. This is to be expected from general arguments (more specifically, the optical theorem): when $q^2 > 4m_f^2$ the fermion-antifermion pair running in the loop can be produced on-shell, generating an imaginary part of the photon self energy.

Let us now go back to the running of the electric coupling. When $-Q^2 \gg m_f^2$ we have

$$e^2(-Q^2) = \frac{e^2(0)}{1 - \frac{e^2(0)}{12\pi^2} \left(\log \frac{Q^2}{m_f^2} - \frac{5}{3} \right)}. \quad (3.51)$$

It is customary to write the variation of of the electric charge with the energy as

$$\frac{de}{d \log Q} = \frac{Q_f^2 e^3}{12\pi^2}. \quad (3.52)$$

This is an example of a Renormalization Group Equation (RGE), which must be solved with the boundary condition that for $Q = 0$ we must obtain $e^2(0)$. More generically, it

can be shown that

$$\frac{de}{d\log Q} \equiv \beta_e = \left(\sum_f \frac{2}{3} Q_f^2 + \sum_s \frac{1}{3} Q_s^2 \right) \frac{e^3}{16\pi^2}, \quad (3.53)$$

where the subscript f denotes **Weyl** fermions and s denotes complex scalars.⁶ This form of the RGE can be computed with any renormalization procedure. It turns out that the $\overline{\text{MS}}$ procedure in which only the terms proportional to $1/\epsilon$ are considered in the counter-terms simplifies the computation drastically. Of course, writing the RGE as in Eq. (3.52) we lose completely any information about the fermion mass. Notice however that expanding Eq. (3.47) for $-Q^2 \ll m_f^2$ we obtain

$$e^2(-Q^2) \simeq e(0)^2 + \frac{e(0)^4}{60\pi^2} \frac{Q^2}{m_f^2}, \quad (3.54)$$

in such a way that even for $Q^2 \sim m_f^2$ we have $e(-Q^2) \stackrel{Q^2 \sim m_f^2}{\simeq} e^2(0)$, at least as long as the coupling e is in the perturbative regime. We thus get to an important conclusion: the electric charge essentially *does not run* below the fermion threshold $Q^2 \sim m_f^2$. This condition can be imposed by hand in the solution to Eq. (3.53), considering the contribution of virtual particles only above its mass.

3.6 Extracting the parameters from data

How is the value of the electric coupling $e(0)$ extracted from experiments? Various techniques are available nowadays, among which measurements in atomic physics and involving the quantum Hall effect. The most precise determination is obtained using Penning traps to infer the anomalous magnetic moment of the electron. The measurement is so precise that we will use it as an input parameter in the comparison between the Standard Model and data in Chapter 8. What is the anomalous magnetic moment of the electron?

In the non-relativistic limit, the Dirac equation in the presence of an electromagnetic field reduces to

$$H = \frac{\mathbf{p}^2}{2M} + V - \frac{e}{2M} \mathbf{B} \cdot (\mathbf{L} + g\mathbf{S}), \quad (3.55)$$

where \mathbf{B} is the magnetic field, \mathbf{L} is the orbital angular momentum and \mathbf{S} is the spin angular momentum. The coupling g is called the Landè coupling, and measures the difference between how the magnetic field couples to the orbital and spin angular momenta. The prediction of the Dirac equation is $g = 2$. To show this we take the non-relativistic (NR) limit of our QED theory. This is done realizing that the NR limit amounts to a $p \ll M$

⁶For a Dirac fermion we thus need to consider both the LH and RH components explicitly in the computation.

limit, in which the total energy of a free particle can be written as $E = M + \mathbf{p}^2/(2M)$, with the second term smaller than the first one. In the time evolution of states we can thus write $\exp(-iEt) \simeq \exp(-iMt - i\mathbf{p}^2t/(2M))$, with the first term the dominant one. It is thus convenient to isolate explicitly this term, and write the QED spinor field as

$$\Psi(x) \rightarrow e^{-iMt}\Psi(x). \quad (3.56)$$

After this substitution, the new $\Psi(x)$ is the NR fermion field. We use capital greek letters for reasons that will become clear momentarily. Another problem that we must face is: what happens to antiparticles? We know that no antiparticles are present in the NR limit because there is not enough energy to excite them. Thus we will have to somehow “lose” some component of the spinor field going in the NR limit. We will see how this happens in a moment.

Let us now apply the NR transformation in Eq. (3.56) to write the QED Lagrangian in terms of the NR field. The Lagrangian results in

$$\mathcal{L}_{NR} = \Psi^\dagger \left[i \frac{\partial}{\partial t} + i\gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + (1 - \gamma^0)M \right] \Psi. \quad (3.57)$$

Inspecting the previous Lagrangian we see that the leading term inside the square brackets is given by the mass term, which is proportional to $(1 - \gamma^0)$. We can now introduce two projectors

$$P_\pm = \frac{1 \pm \gamma^0}{2}, \quad (3.58)$$

completely analogous to the chirality projectors $P_{L/R}$, and write $\Psi = P_+\Psi + P_-\Psi \equiv \psi + \chi$. Using the algebra of the Dirac matrices, the NR Lagrangian above can be written in terms of the χ and ψ fields as follows:

$$\mathcal{L}_{NR} = \chi^\dagger \left[i \frac{\partial}{\partial t} + 2M \right] \chi + i\psi^\dagger \frac{\partial}{\partial t} \psi + i [\psi^\dagger \gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} \chi + \chi^\dagger \gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} \psi]. \quad (3.59)$$

We can understand what is happening looking at the order of magnitude of the different terms. In the first square bracket the first term is of the order of the kinetic energy of the particle associated with χ , and is thus much smaller than the mass term because we are in the NR regime. All the other terms involve either time or space derivatives, and are thus either of the order of the momentum or of the energy. The term linear in $2M$ is thus the dominant one, and is associated with χ . This means that χ is heavier than ψ , and can be *integrated out* at energies $E \ll M$. This is connected to the idea of EFT already discussed: when $E \ll M$ there is not enough energy to excite the χ particle, and we can thus eliminate it from the Lagrangian. In order to integrate out χ in a consistent way we can invoke path integral techniques, which tell us that we must compute the equations of motion for χ and substitute them back in the Lagrangian in order for χ to completely

disappear from the theory in a consistent way. The EoM are:

$$\left[i \frac{\partial}{\partial t} + 2M \right] \chi + i \gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} \psi = 0 \Rightarrow \chi = \frac{-i \gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} \psi}{i \frac{\partial}{\partial t} + 2M} \simeq \frac{-i \gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} \psi}{2M} + \dots \quad (3.60)$$

where in the last step we have expanded in $1/M$, since M is a large scale in the NR limit. Inserting this solution in the Lagrangian and considering only terms up to $\mathcal{O}(M^{-1})$ one obtains

$$\mathcal{L}_{NR} = \psi^\dagger \left[i \frac{\partial}{\partial t} + \frac{\boldsymbol{\nabla}^2}{2M} \right] \psi + \dots \quad (3.61)$$

Notice that we obtained precisely the Lagrangian for a quantum Schroedinger field, see Eq. (1.26). In order to make contact with the Hamiltonian of the Dirac equation in an external electromagnetic field we switch on the QED coupling with a photon $A^\mu = (\Phi, \mathbf{A})$ and repeat the previous deduction. The EoM now reads

$$\chi = \frac{-i \gamma^0 \boldsymbol{\gamma} \cdot [\boldsymbol{\nabla} - ie \mathbf{A}] \psi}{i \frac{\partial}{\partial t} + 2M} \simeq \frac{-i \gamma^0 \boldsymbol{\gamma} \cdot [\boldsymbol{\nabla} - ie \mathbf{A}] \psi}{2M} + \dots \quad (3.62)$$

For the previous expansion to make sense we also need the typical scale of the EM field to be much smaller than the fermion mass M . Substituting back in the Lagrangian we need to pay attention to the fact that the operator $\mathbf{D} \equiv \boldsymbol{\nabla} - ie \mathbf{A}$ does not commute with itself because of the derivative. The Lagrangian results in

$$\mathcal{L} = \psi^\dagger \left[i \frac{\partial}{\partial t} - \frac{\gamma^i \gamma^k D_i D_k}{2M} \right] \psi. \quad (3.63)$$

We now write $\gamma^i \gamma^k$ in terms of the commutator and anticommutator to separate the symmetric and antisymmetric contributions. The symmetric one is simplified observing that the algebra of the Dirac matrices amounts to

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \{\gamma^i, \gamma^j\} = -2\delta^{ij}. \quad (3.64)$$

As for the antisymmetric part, a simple computation shows that we can write

$$[\gamma^i, \gamma^j] = -4i\epsilon^{ijk} S^k, \quad \mathbf{S} = \frac{1}{2} \left(\begin{array}{c|c} \boldsymbol{\sigma} & 0 \\ \hline 0 & \boldsymbol{\sigma} \end{array} \right), \quad (3.65)$$

where we have defined the spin operator \mathbf{S} . The last ingredient we need is the antisymmetric combination of the (spatial) covariant derivatives:

$$D_i D_k - D_k D_i = -ie (\partial_i A^k - \partial_k A^i) = ie F_{ik}. \quad (3.66)$$

In the last step we have used the fact that $F_{ik} = \partial_i A_k - \partial_k A_i = -(\partial_i A^k - \partial_k A^i)$ because of the lower position of the indices. Finally, we report here the contraction

$$[\gamma^i, \gamma^j] F_{ij} = 8i \mathbf{S} \cdot \mathbf{B}, \quad (3.67)$$

where \mathbf{B} is the magnetic field. Putting all together we obtain

$$\begin{aligned}
 \gamma^i \gamma^k D_i D_k &= \left(\frac{\{\gamma^i, \gamma^k\}}{2} + \frac{[\gamma^i, \gamma^k]}{2} \right) D_i D_k \\
 &= -\mathbf{D}^2 + \frac{[\gamma^i, \gamma^k]}{2} \frac{D_i D_k - D_k D_i}{2} \\
 &= -\mathbf{D}^2 + \frac{[\gamma^i, \gamma^k] (ieF_{ik})}{4} \\
 &= \mathbf{D}^2 - 2e\mathbf{S} \cdot \mathbf{B}.
 \end{aligned} \tag{3.68}$$

The NR Lagrangian finally results in

$$\begin{aligned}
 \mathcal{L} &= \psi^\dagger \left[i \frac{\partial}{\partial t} - \frac{\gamma^i \gamma^k D_i D_k}{2M} \right] \psi \\
 &= \psi^\dagger \left[i \frac{\partial}{\partial t} + \frac{\mathbf{D}^2}{2M} + \frac{e}{M} \mathbf{B} \cdot \mathbf{S} \right] \psi.
 \end{aligned} \tag{3.69}$$

Comparing with Eq. (3.55) we see that, as anticipated, $g = 2$.


What type of terms contribute, in general, to the magnetic moment of fermions? Since we must search for couplings between the external magnetic field and the system spin, the obvious choice is to look for operators that involve the field strength $F_{\mu\nu}$ (since the field strength contains directly the magnetic –and electric – fields). The smallest dimensional operator is thus

$$\mathcal{O}_{dipole} = \bar{\Psi} \sigma^{\mu\nu} \Psi F_{\mu\nu}, \quad \sigma^{\mu\nu} = \frac{i [\gamma^\mu, \gamma^\nu]}{2}. \tag{3.70}$$

In the NR limit this operator simplifies to

$$\mathcal{O}_{dipole}^{NR} = 2i (\psi^\dagger \boldsymbol{\gamma} \psi) \cdot \mathbf{E} - 4 (\psi^\dagger \mathbf{S} \psi) \cdot \mathbf{B} \tag{3.71}$$

and thus, as expected, there is a contribution to the magnetic moment of the fermion. The first term is instead a coupling between the fermion current and the electric field.

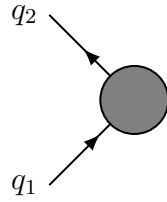
 **Exercise 3.2** Derive the previous result.

The question is now: do quantum corrections modify the tree-level value of the magnetic moment? The answer is yes, and as we are going to see, the correction will be proportional to

$$\alpha_e \equiv \frac{e^2}{4\pi}, \tag{3.72}$$

allowing to measure the electric charge from the measurement of the magnetic moment of the electron. In fact, what radiative corrections in QED do is to generate the dipole operator, and thus corrections to the magnetic dipole moment. The radiative corrections that will give us this contribution are depicted in the last diagram of Fig. 3.1. The most

general form of this 1PI diagram is



$$= \bar{u}_2 (f_1 \gamma^\mu + f_2 q_1^\mu + f_3 q_2^\mu) u_1, \quad (3.73)$$

where $f_{1\dots 3}$ may be functions of invariant products of the momenta. Notice that we did not write a term linear in $p = q_2 - q_1$ because this would not be independent from the terms with $q_{1,2}^\mu$ because of momentum conservation. Moreover, we can use gauge invariance to simplify further the expression. To this end, let us suppose that the fermions are on-shell, so that we can apply

$$q_1 u_1 = m u_1, \quad \bar{u}_2 q_2 = m \bar{u}_2. \quad (3.74)$$

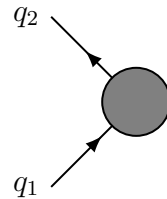
Keeping the photon off-shell (in such a way that the kinematics is consistent) we obtain

$$\begin{aligned} p_\mu \bar{u}_2 (f_1 \gamma^\mu + f_2 q_1^\mu + f_3 q_2^\mu) u_1 &= \bar{u}_2 \not{p} u_1 + (f_2 p \cdot q_1 + f_3 p \cdot q_2) \bar{u}_2 u_1 \\ &= m \bar{u}_2 u_1 - m \bar{u}_2 u_1 + (f_2 p \cdot q_1 + f_3 p \cdot q_2) \bar{u}_2 u_1 \\ &= (f_2 p \cdot q_1 + f_3 p \cdot q_2) \bar{u}_2 u_1. \end{aligned} \quad (3.75)$$

Kinematics tells us that

$$p \cdot q_1 = -\frac{p^2}{2} = -p \cdot q_2, \quad (3.76)$$

which inserted in the previous equation allows to conclude that $f_2 = f_3$. We can thus write



$$= \bar{u}_2 (f_1 \gamma^\mu + f_2 (q_1^\mu + q_2^\mu)) u_1, \quad (3.77)$$

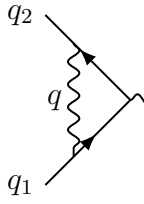
The last piece of information we need is the Gordon identity, valid for on-shell fermions:

$$\bar{u}_2 \gamma^\mu u_1 = \frac{1}{2m} \left((q_1 + q_2)^\mu \bar{u}_2 u_1 + i p_\alpha \bar{u}_2 \sigma^{\mu\alpha} u_1 \right). \quad (3.78)$$

This identity allows us to express the terms proportional to the momenta in terms of $\sigma^{\mu\nu}$ (the Dirac structure needed to understand what the loop contribution to the magnetic moment is) and the fermion momenta. This separation of terms makes clear that the vertex will have two terms: (i) a term proportional to γ^μ : the corrections to this term will enter in the definition of the renormalized electric charge; (ii) a term proportional to $\sigma^{\mu\nu}$ and hence to the fermion spin, that corresponds to the dipole moments. Moreover, since the measurement is done at very low energy, we will only worry to compute the

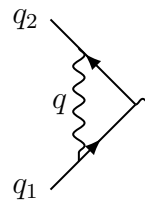
vertex function in the $p \rightarrow 0$ limit.

At the 1-loop level the only diagram we need to worry about is




$$\begin{aligned}
 &= (-ieq)^3 \int \frac{d^4 q}{(2\pi)^4} \bar{u}_2 \gamma^\mu \frac{i(\not{q} + \not{q}_2 + M)}{(q + q_2)^2 - M^2} \gamma^\alpha \frac{i(\not{q} + \not{q}_1 + M)}{(q + q_1)^2 - M^2} \gamma^\nu u_1 \\
 &\quad - \frac{i}{q^2} \left(g^{\mu\nu} - (1 - \xi) \frac{q^\mu q^\nu}{q^2} \right).
 \end{aligned} \tag{3.79}$$

A straightforward but tedious computation gives



$$p \rightarrow 0 = \frac{e^3}{16\pi^2 M} \bar{u}_2 \sigma^{\alpha\beta} u_1 p_\beta + \dots, \tag{3.80}$$

where the dots represent terms containing γ^μ (both finite and divergent) that, as we already said, are important for the renormalization of the electric charge.

 **Exercise 3.3** Do explicitly the 1-loop computation above using Package-X.

It is easy to see that the amplitude just computed can be represented by the following additional term in the Lagrangian:

$$\Delta\mathcal{L} = -\frac{e^3}{32\pi^2 M} \bar{\Psi} \sigma^{\mu\nu} \Psi F_{\mu\nu}, \tag{3.81}$$

which is exactly of the form we were looking for. Combining this result with Eq. (3.71) we finally obtain

$$\mathcal{L}_{NR} = \psi^\dagger \left[i \frac{\partial}{\partial t} + \frac{\mathbf{D}^2}{2M} + \frac{e}{2M} \left(2 + \frac{\alpha}{\pi} \right) \mathbf{B} \cdot \mathbf{S} \right] \psi. \tag{3.82}$$

The loop contribution thus forces the magnetic moment to deviate from its tree-level value and is thus called *anomalous* magnetic moment. We are going to come back to the measurement of α via the anomalous magnetic moment of the electron in Chapter 8.

3.7 Higher dimensional terms

We conclude this chapter with some comments about the effect of higher dimensional operators. As we already discussed many times, the Wilsonian picture of QFT dictates that the renormalizable theory of Eq. (3.1) just displays the terms relevant in the $E \ll \Lambda$ limit. As we move to higher energy, additional terms suppressed by increas-

ing powers of Λ must be considered. Alternatively, if these higher dimensional terms generate physical effects not generated by the renormalizable Lagrangian we can hope to detect them, opening a window on the Wilson coefficients of such higher dimensional terms.

In this section we are going to focus on two effects generated by higher dimensional operators:

- The dimension-5 operator

$$\frac{c_{MDM}}{\Lambda} \bar{\Psi} \sigma^{\mu\nu} \Psi F_{\mu\nu} \quad (3.83)$$

gives an additional contribution to the magnetic dipole moment. If this contribution is comparable to QED loop corrections then it would modify the value of α extracted from experiments, changing other predictions;

- The dimension-5 operator

$$\frac{c_{EDM}}{\Lambda} \bar{\Psi} \sigma^{\mu\nu} \gamma_5 \Psi F_{\mu\nu} \quad (3.84)$$

gives a contribution to the *electron dipole moment*. To understand why the former operator contributes to the magnetic dipole while the latter contributes to the latter it is useful to write the explicit form of the Dirac matrices in the two cases:

$$\begin{aligned} \gamma^0 \sigma^{\mu\nu} F_{\mu\nu} &= \begin{pmatrix} 0 & 2i\boldsymbol{\sigma} \cdot \mathbf{E} \\ -2i\boldsymbol{\sigma} \cdot \mathbf{E} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2\boldsymbol{\sigma} \cdot \mathbf{B} \\ -2\boldsymbol{\sigma} \cdot \mathbf{B} & 0 \end{pmatrix} \\ &= 2i\boldsymbol{\gamma} \cdot \mathbf{E} - 4\mathbf{S} \cdot \mathbf{B}, \\ \gamma^0 \sigma^{\mu\nu} F_{\mu\nu} \gamma_5 &= \begin{pmatrix} 0 & 2i\boldsymbol{\sigma} \cdot \mathbf{E} \\ 2i\boldsymbol{\sigma} \cdot \mathbf{E} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2\boldsymbol{\sigma} \cdot \mathbf{B} \\ 2\boldsymbol{\sigma} \cdot \mathbf{B} & 0 \end{pmatrix}, \\ &= 4i\gamma^0 \mathbf{S} \cdot \mathbf{E} - 2\boldsymbol{\gamma} \cdot \mathbf{B}. \end{aligned} \quad (3.85)$$

The last expression has been obtained comparing the matrix structures in the two cases. As we see, a magnetic dipole operator generates a coupling between the magnetic field and the spin of the fermion, while the electric dipole operators generates a coupling between the electric field and the spin of the fermion.


The electric dipole operator is important because it violates the CP (charge conjugation/parity) symmetry of the renormalizable QED Lagrangian. The parity and charge conjugation transformations are

$$\begin{array}{ll} \text{Parity} & \text{Charge conjugation} \\ \psi \rightarrow \gamma^0 \psi & \psi \rightarrow i\gamma^0 \gamma^2 \bar{\psi}^T \\ A_\mu \rightarrow (-1)^\mu A_\mu & A_\mu \rightarrow -A_\mu \end{array} \quad (3.86)$$

where $(-1)^\mu = 1$ for $\mu = 0$ and $(-1)^\mu = -1$ for $\mu = i$. The total effect on useful quantities appearing in the Lagrangian is summarized in Tab. 3.1.

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	$\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi$	∂_μ
P	+1	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$-(-1)^\mu(-1)^\nu$	$(-1)^\mu$
C	+1	+1	-1	+1	-1	-1	+1

Table 3.1: Transformation of useful quantities under parity and charge conjugation.

 **Exercise 3.4** Derive explicitly the results of the table.

It is easy to see that the renormalizable QED Lagrangian is invariant under the combined action of C and P (an operation called CP). At the level of $d = 5$ operators we instead have

$$\begin{aligned}\bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu} &\rightarrow [-(-1)^\mu(-1)^\nu]^2 \bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu} = \bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu} \\ \bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu} &\rightarrow - [(-1)^\mu(-1)^\nu]^2 \bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu} = -\bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu}\end{aligned}\quad (3.87)$$

We see that the electric dipole operator violates P and hence CP , unlike all other terms. This computation elucidates two points: (i) as already discussed, the low energy Lagrangian has more symmetry than the higher order terms, and (ii) since the symmetry breaking effect is absent at the renormalizable level, we can search for it experimentally. An electron dipole moment has been searched extensively, with no positive result found. To be compatible with observations (especially the recent measurement from the ACME collaboration) the coefficient in front of the electric dipole operators must satisfy

$$\frac{c_{EDM}}{\Lambda} \lesssim 10^{-29} e \text{ cm} \simeq 3 \times 10^{-19} \frac{e}{m_e}. \quad (3.88)$$

This bound implies that either Λ is a very large scale, or that for some reason the coefficient c_{EDM} is very small. Either way, the very fact that the $d = 5$ operator produces a physical effect that is not predicted by the $d \leq 4$ terms allows to put very strong bounds on the coefficient of such an operator.

3.8 Additional readings

- S.Weinberg, “The Quantum Theory of Fields” vol. 1;
- M.Peskin and D.Schroeder, “An Introduction To Quantum Field Theory”;
- S.Coleman, “Lectures on Quantum Field Theory”;
- M.Schwartz, “Quantum Field Theory and the Standard Model”;
- R. Penco, “An Introduction to Effective Field Theories” ([link](#));

Part II

Strong interactions: Quantum Chromodynamics

Chapter 4 The quark model and $SU(3)$

4.1 Mesons and baryons properties

We start our discussion of strong interactions reminding the reader about the observed structure of hadrons. Along the first half of the XX century a plethora of new particles were discovered and some basic facts emerged:

- hadrons can be distinguished between *mesons* (either with spin 0 or spin 1) and *baryons* (spin 1/2 or 3/2). Typically the baryons are much heavier than the spin 0 mesons, and somewhat heavier than the vector mesons;
- the light spin 0 mesons are pseudoscalar, i.e. they are odd under a parity transformation;
- the light pseudoscalar mesons are (in order of increasing mass): three pions (π^0 and π^\pm) with masses around 140 MeV; four kaon states (the neutral K^0 and \bar{K}^0 and the charged K^\pm) with masses around 500 MeV; the η particle, with mass around 550 MeV. Given the similarity between their masses, we will consider them as a unique pseudoscalar octet;
- in reactions involving hadrons it was discovered a new conserved quantum number, the *strangeness*. A diagram with a classification of the pseudoscalars in terms of their electric charge and strangeness is shown in Fig. 4.1;
- the spin 1/2 baryons are (in order of increasing mass): neutron (n) and proton (p), with mass around 938 MeV; Λ^0 , with mass around 1116 MeV; the Σ system (Σ^0 and Σ^\pm), with mass around 1192 MeV; the Ξ system (Ξ^0 and Ξ^-), with mass

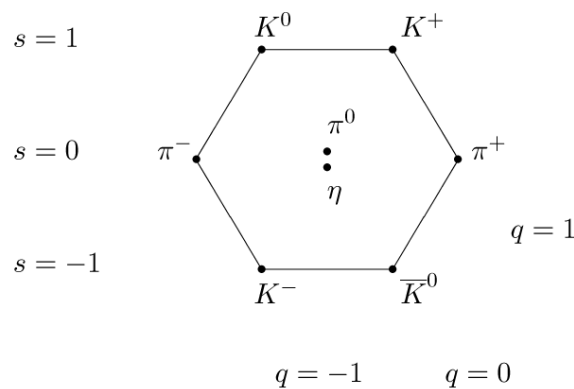


Figure 4.1: Classification of the light pseudoscalar mesons in terms of their electric charge q and of their strangeness s .

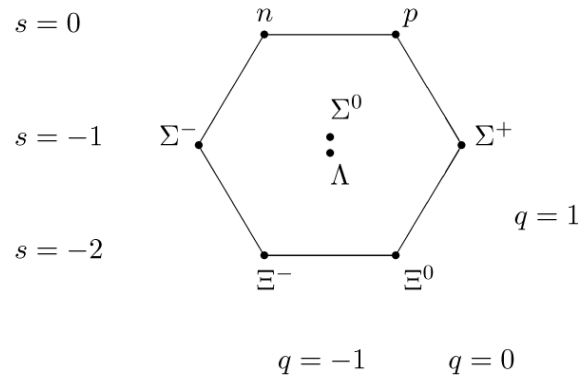


Figure 4.2: Classification of the spin 1/2 baryons in terms of their electric charge q and of their strangeness s .

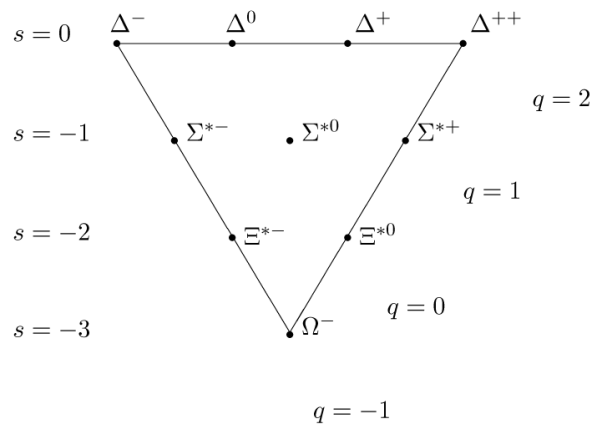


Figure 4.3: Classification of the spin 3/2 baryons in terms of their electric charge q and of their strangeness s .

around 1315 MeV. One more time, given the similarity between their masses, we will consider the system of spin 1/2 baryons as a unique baryon octet;

- a classification in terms of electric charge and strangeness can be given also for the spin 1/2 baryons. It is shown in Fig. 4.2;
- the spin 3/2 baryons are (again in order of increasing mass): the Δ system (Δ^- , Δ^0 , Δ^+ and Δ^{++}), with masses around 1232 MeV; the Σ^* system (Σ^{*-} , Σ^{*0} and Σ^{*+}) with masses around 1385 MeV; the Ξ^* system (Ξ^{*-} and Ξ^{*0}) with masses 1532 MeV; the Ω^- , with mass 1672 MeV. We will consider them as organized in a decuplet;
- the classification of the spin 3/2 baryons in terms of their electric charge and strangeness is shown in Fig. 4.3.

A rationale behind these seemingly unrelated properties came in 1961 when Gell-Mann and Ne'eman proposed to interpret mesons and baryons as bound states of elementary *quarks*. The idea is as follows: suppose there exist more fundamental degrees of freedom

u , d and s , whose bound states organize themselves in the mesons and baryon octets and in the baryon decuplet. Notice that since mesons and baryons are bosons and fermions, respectively, the quarks cannot be scalar particles, but must be fermions. Let us discuss a bit more in detail the situation, since it will justify our subsequent findings. Since the quarks are spin $1/2$ particles, the spin of their bound states can simply be computed combining many spin $1/2$ representations of the rotation group. The result is well known from Quantum Mechanics: in terms of spin and dimensions¹ we have

$$\frac{1}{2} \times \frac{1}{2} = 0_A + 1_S, \quad \mathbf{2} \times \mathbf{2} = \mathbf{1}_A + \mathbf{3}_S. \quad (4.1)$$

The subscripts A and S indicate the antisymmetric and symmetric combinations, respectively. The result can easily be derived using the techniques familiar from the Quantum Mechanics courses. We will now take a detour to discuss a generic *tensor method* to understand these results.

4.2 Detour: tensor methods in group theory

It is well known that group theory plays a fundamental role in the discussion of symmetries in Physics. A reach subject is the one of the representations of a group. We can summarize the situation as follows: consider a certain group element $g \in \mathcal{G}$. The group \mathcal{G} , as well as the element g , are abstract object at this point. It is useful to find concrete expressions for the elements g (for instance, in terms of matrices). When we do this we say that we are searching for *representations* of the group. Let us focus on matrix representations from now on, since they are those that will play a major role in what follows. The dimension of the space on which the matrices of the representation are acting is called the dimension of the representation. For instance, the spin 1 representation in Eq. (4.1) is acting on a 3 dimensional space and can be represented by a 3×3 matrix.

Let us now specialize our discussion to the $SU(N)$ groups. They are defined as the set of matrices U such that

$$U^\dagger U = \mathbf{1}, \quad \det U = 1. \quad (4.2)$$

The representation in a N -dimensional space is called “fundamental” representation, and consists of the U matrices themselves. We will denote it with \mathbf{N} . Consider now the case in which two set of matrices $\{A\}$ and $\{B\}$ give a representation of the group. If

¹We remind the reader that a spin j representation has dimension $2j + 1$.

there exist an invertible matrix P such that

$$B = P^{-1} A P \quad (4.3)$$

then the two representations are *equivalent* and they count as a unique representation. This is the case of the complex conjugate representation $\bar{\mathbf{N}}$ given by the matrices $\{U^*\}$ when $N = 2$. We thus write $\bar{\mathbf{2}} \sim \mathbf{2}$ (for $SU(2)$) to denote that the two representations are equivalent. For all $N \geq 3$, the complex conjugate representation $\bar{\mathbf{N}}$ is instead a representation independent from \mathbf{N} .

The last concept we need for the moment is the one of *irreducible* representation. We will not be interested in the precise mathematical definition, rather on understanding what this means. To be concrete, we will consider an example taken from $SU(2)$: the representation $\mathbf{4}$. As we know from Eq. (4.1), we can decompose $\mathbf{4} = \mathbf{1}_A + \mathbf{3}_S$. What does this mean? It means that if we take an object in the 4-dimensional space on which $\mathbf{4}$ is acting and we apply an $SU(2)$ transformation, the 4 components of the object will in general transform one in the other. However, there is a basis of the space in which the 4×4 matrix U_4 can be decomposed as

$$U_4 = \left(\begin{array}{c|c} U_3 & 0 \\ \hline 0 & U_1 \end{array} \right). \quad (4.4)$$

We see that in this basis the space on which the $\mathbf{4}$ is acting is “splitted” into a 3-dimensional space and a 1-dimensional space, and the two subspaces do not mix with each other under the action of U_4 . This decomposition cannot be further extended, and we say that $\mathbf{3}$ and $\mathbf{1}$ are *irreducible* representations.

Our task will now be to describe one way to obtain such irreducible representations. Let us start with a tensor $u_i, i = 1, \dots, N$, transforming in the fundamental \mathbf{N} . Under a $SU(N)$ transformation the tensor transforms as

$$u_i \rightarrow U_{ij} u_j. \quad (4.5)$$

Its complex conjugate instead transforms as

$$u_i^* \rightarrow U_{ij}^* u_j^* = u_j^* (U^\dagger)_{ji} \quad (4.6)$$

Let us now introduce some notation that will simplify the index manipulation. We define raised and lowered indices as

$$U_{ij} \equiv U_i^j, \quad U_{ij}^* \equiv \bar{U}_j^i, \quad u_i^* \equiv \bar{u}^i. \quad (4.7)$$

With this notation the tensor transformations become

$$u_i \rightarrow U_i^j u_j, \quad \bar{u}^i \rightarrow \bar{U}_j^i \bar{u}^j, \quad (4.8)$$

i.e. the order of the indices is exactly the one expected. The unitary conditions $U^\dagger U =$

$\mathbf{1} = UU^\dagger$ become

$$\bar{U}_i^k U_k^m = \delta_i^m, \quad U_i^k \bar{U}_k^m = \delta_i^m. \quad (4.9)$$

The fact that the determinant of $SU(N)$ matrices is unity can be expressed using the identity

$$\epsilon^{i_1 \dots i_N} U_{i_1}^{j_1} \dots U_{i_N}^{j_N} = \det U \epsilon^{j_1 \dots j_N} = \epsilon^{j_1 \dots j_N}, \quad (4.10)$$

where $\epsilon^{i_1 \dots i_N}$ is the Levi-Civita tensor with N indices. The same identity applies also for the conjugate representation. Using Eqs. (4.9) and (4.10) we immediately conclude that the δ and ϵ symbols are invariant under $SU(N)$ transformations.

Let us now take the 2-tensor $u_i u_j$. Under an $SU(N)$ transformation it transforms according to the product

$$u_i u_j \rightarrow U_{ik} U_{jm} u_k u_m. \quad (4.11)$$

This is a representation of $SU(N)$ of dimension N^2 , which we will call $\mathbf{N} \times \mathbf{N}$. How do we find the irreducible representations embedded in $\mathbf{N} \times \mathbf{N}$? A general method is to try to decompose as much as possible the tensor in terms of the invariant objects, i.e. δ_j^i and $\epsilon_{i_1 \dots i_N}$. Notice that for the first we have a ‘‘covariant’’ and a ‘‘contravariant’’ index, while for the ϵ symbols all the indices are either covariant or contravariant.


Let us discuss what happens in a simple example for which we already know the answer, i.e. $\mathbf{2} \times \mathbf{2}$ in $SU(2)$. From the tensor point of view we write

$$u_i v_j = \frac{1}{2} \epsilon_{ij} \epsilon^{mn} u_m v_n + \left(u_i v_j - \frac{1}{2} \epsilon_{ij} \epsilon^{mn} u_m v_n \right). \quad (4.12)$$

The first term is the antisymmetric combination of the 4 independent entries of u_i and v_j , and it amounts to 1 combination. The combination between brackets is completely symmetric, and contains thus 3 independent combinations of the parameters. We thus recover

$$\mathbf{2} \times \mathbf{2} = \mathbf{1}_A + \mathbf{3}_S \quad (4.13)$$

as expected.

 **Exercise 4.1** Show that the combination between brackets in Eq. (4.12) is the symmetric combination.

Notice that the decomposition in symmetric/antisymmetric components is always pos-

sible, independently from N . For instance, for $SU(3)$ we obtain

$$\begin{aligned} u_i v_j &= \frac{1}{2} \epsilon_{ijk} \epsilon^{kmn} u_m v_n + \left(u_i v_j - \frac{1}{2} \epsilon_{ijk} \epsilon^{kmn} u_m v_n \right) \\ &= A_{ij} + S_{ij} . \end{aligned} \quad (4.14)$$

The combination between brackets has dimension 6, being the symmetric combination of two objects with three entries each. As for the first combination, we know that it must be of dimension 3. In $SU(3)$ we however have two inequivalent 3-dimensional representations, the fundamental $\mathbf{3}$ and the antifundamental $\bar{\mathbf{3}}$. Which one of the two appears here? To understand this point, let us apply $SU(3)$ transformations:

$$\epsilon_{ijk} \epsilon^{kmn} u_m v_n \rightarrow \epsilon_{ijk} \epsilon^{kmn} U_m^r U_n^s u_r v_s . \quad (4.15)$$

We now use Eq. (4.10) to write

$$\epsilon^{vmn} U_m^r U_n^s = \bar{U}_t^v \epsilon^{trs} , \quad (4.16)$$

and we finally obtain

$$\epsilon_{ijk} \epsilon^{kmn} u_m v_n \rightarrow \epsilon_{ijk} \epsilon^{kmn} U_m^r U_n^s u_r v_s = \bar{U}_t^k \epsilon_{ijk} \epsilon^{trs} u_r v_s . \quad (4.17)$$

As we see the central object is the combination $\epsilon^{kmn} u_m v_n \equiv z^k$, from which we conclude that

$$\epsilon^{kmn} u_m v_n \sim \bar{\mathbf{3}} . \quad (4.18)$$

Putting all together we finally obtain

$$\mathbf{3} \times \mathbf{3} = \mathbf{6} + \bar{\mathbf{3}} . \quad (4.19)$$

Notice that in $SU(3)$ (and only for this group) the antifundamental can be represented by either

$$A_{ij} \quad \text{or} \quad z^k , \quad (4.20)$$


with the two representations connected by a Levi-Civita tensor via

$$A_{ij} = \epsilon_{ijk} z^k . \quad (4.21)$$

What happens if we multiply one fundamental and one antifundamental representation? In this case we cannot use the Levi-Civita tensor for the decomposition, but we must use the Kronecker delta, since it has the right combination of indices. More specifically, for a generic $SU(N)$ we write

$$\begin{aligned} u_i v^j &= \frac{1}{N} \delta_i^j (v^k u_k) + \left(u_i v^j - \frac{1}{N} \delta_i^j (v^k u_k) \right) \\ &= \delta_i^j S + (\text{Ad})_i^j . \end{aligned} \quad (4.22)$$

The first term contains a unique (invariant) combination of u and v , and it has thus dimension 1. It is called the singlet S .

 **Exercise 4.2** Show that the singlet S does not transform.

The second term has dimension $N^2 - 1$. It is called the adjoint representation Ad . For $SU(3)$ this implies

$$\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8} . \quad (4.23)$$

It is important to notice that the adjoint representation transforms as

$$(\text{Ad})_j^i \rightarrow U_j^k \bar{U}_m^i (\text{Ad})_k^m = U_{jk} (\text{Ad})_{km} U_{im}^* = [U(\text{Ad})U^\dagger]_{ji} \Rightarrow (\text{Ad}) \rightarrow U(\text{Ad})U^\dagger . \quad (4.24)$$

We can now move on considering higher order tensors. The computation quickly becomes rather cumbersome, although systematic and general techniques can be developed to deal with such situations. For some example, we refer the reader to [these lectures](#). Let us however show one example that will be important for us, i.e. the product of three fundamentals of $SU(3)$. We have

$$u_i v_j w_k = u_i (A_{jk} + S_{jk}) \Leftrightarrow \mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{3} \times (\bar{\mathbf{3}} + \mathbf{6}) . \quad (4.25)$$

We have already worked out the product $\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8}$: defining $z^k \equiv \epsilon^{kmn} v_m w_n / 2$ we have

$$u_i A_{jk} = u_i \epsilon_{jkm} z^m = \epsilon_{jkm} \left[\left(u_i z^m - \frac{1}{3} \delta_i^m (z^r u_r) \right) + \frac{1}{3} \delta_i^m (z^r u_r) \right] , \quad (4.26)$$

where again the first term is the $\mathbf{8}$ representation while the second term is the $\mathbf{1}$. We notice that $\mathbf{1}$ is the completely antisymmetric combination, while the octet $\mathbf{8}$ is antisymmetric in two of the indices.

As for $\mathbf{3} \times \mathbf{6}$, we try to decompose the product $u_i S_{jk}$ in terms of Levi-Civita tensors as much as possible:

$$u_i S_{jk} = \frac{1}{3} \epsilon_{ijm} \epsilon^{mnp} u_n S_{pk} + \frac{1}{3} \epsilon_{ikm} \epsilon^{mnp} u_n S_{jp} + \left(u_i S_{jk} - \frac{1}{3} \epsilon_{ijm} \epsilon^{mnp} u_n S_{pk} - \frac{1}{3} \epsilon_{ikm} \epsilon^{mnp} u_n S_{jp} \right) . \quad (4.27)$$

The combination between brackets is completely symmetric in the three indices i, j and k , and has thus $N(N+1)(N+2)/3! = 10$ components. The remaining two combinations can be written in the form $\epsilon_{ijm} (\text{Ad})_k^m$ and are thus another way to write the adjoint representation. Notice that the $\mathbf{8}$ representation we have just obtained is symmetric in two of the indices, unlike the one we found above. This will prove important in Section 4.4. Putting all together we have

$$u_i v_j w_k = \epsilon_{ijk} S + \epsilon_{jkm} A_i^m + (\epsilon_{ijm} B_k^m + \epsilon_{ikm} B_j^m) + S_{ijk} , \quad (4.28)$$


where

$$\begin{aligned}
 S &= \frac{1}{3} z^r u_r, \\
 A_i^m &= u_i z^m - \frac{1}{3} \delta_i^m (z^k u_k), \\
 B_k^m &= \frac{1}{3} \epsilon^{mnp} u_n S_{pk}, \\
 S_{ijk} &= u_i S_{jk} - \frac{1}{3} \epsilon_{ijm} \epsilon^{mnp} u_n S_{pk} - \frac{1}{3} \epsilon_{ikm} \epsilon^{mnp} u_n S_{jp},
 \end{aligned}
 \tag{4.29}$$


where we remind once more that A_i^m is antisymmetric under $v \leftrightarrow w$ while B_k^m is symmetric under the same exchange. This can be schematically written as

$$\mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{1}_A + \mathbf{8}_{M_A} + \mathbf{8}_{M_S} + \mathbf{10}_S.
 \tag{4.30}$$

In the formula above M_A denotes the ‘‘mixed antisymmetric’’ combination (antisymmetric in two of the indices) while M_S denotes the ‘‘mixed symmetric’’ combination.

 **Exercise 4.3** Show that the singlet combination appearing in the $\mathbf{3} \times \mathbf{3} \times \mathbf{3}$ product is

$$\epsilon^{ijk} u_i v_j w_k.$$

 **Exercise 4.4** Compute explicitly the expression of the $\mathbf{8}_{M_A}$ and $\mathbf{8}_{M_S}$ representations in terms of the components of u , v and w .

4.3 The eightfold way and isospin

Let us now show why the previous discussion about group theory is relevant for the description of mesons and baryons. We recall that the pseudoscalar mesons appear to form an octet, while baryons come either in an octet (for spin 1/2) or in a decuplet (for spin 3/2). We can thus (with Gell-Mann and Ne’eman) interpret the results as follows:

- Mesons can appear in an octet if they are bound states of a quark-antiquark pair $\bar{q}q$, transforming respectively as a $\mathbf{3}$ and $\bar{\mathbf{3}}$ of $SU(3)$;
- Baryons can appear in octets and decuplet if they are bound states of three quarks qqq ;

the triplet of quarks is denoted by

$$q = (u, d, s)^T, \quad (4.31)$$

with u = up quark, d = down quark and s = strange quark. The $SU(3)$ under which $q \sim \mathbf{3}$ is called “flavor” $SU(3)_F$ (not to be confused with the “color” $SU(3)_c$ that we will encounter soon). Were the flavor $SU(3)_F$ an exact symmetry of the Hamiltonian then the members of the octet and decuplet would be exactly degenerate. Since this is not observed in nature (the masses of the members of the multiplets are slightly different) we conclude that $SU(3)_F$ is only an approximate symmetry of nature. We will come back later on the origin of the breaking.

Let us now discuss how to construct the adjoint matrix of mesons looking at their quark constituents. According to Eq. (4.22) the pseudoscalar meson octet corresponds to

$$\Pi_i^k = \bar{q}^k q_i - \frac{1}{3} \delta_i^k (\bar{q}q) = \begin{pmatrix} \frac{2u\bar{u}-d\bar{d}-s\bar{s}}{3} & u\bar{d} & u\bar{s} \\ d\bar{u} & \frac{-u\bar{u}+2d\bar{d}-s\bar{s}}{3} & d\bar{s} \\ s\bar{u} & s\bar{d} & \frac{-u\bar{u}-d\bar{d}+2s\bar{s}}{3} \end{pmatrix}_i^k. \quad (4.32)$$

The off-diagonal mesons are defined as $\pi^+ \equiv u\bar{d}$, $\pi^- \equiv d\bar{u}$, $K^+ = u\bar{s}$, $K^- = s\bar{u}$, $K^0 = d\bar{s}$ and $\bar{K}^0 = s\bar{d}$. With this choice we immediately see that the electric charge of the quarks are

$$q_u = \frac{2}{3}, \quad q_d = -\frac{1}{3}, \quad q_s = -\frac{1}{3}. \quad (4.33)$$


For the diagonal components some care must be taken, since the tracelessness condition implies that only two independent combinations are present. The established choice is

$$\pi^0 = \frac{u\bar{u} - d\bar{d}}{\sqrt{2}}, \quad \eta = \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}}. \quad (4.34)$$

In terms of these combinations we finally obtain the meson matrix written as

$$\Pi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{pmatrix}. \quad (4.35)$$

Notice that this matrix is hermitian.

 **Exercise 4.5** Show that the matrix Π has exactly the form shown in terms of the mesons defined in Eq. (4.34).

Inspecting the experimental masses of the quarks we see that the strange quark is much heavier than the up and down quarks. This means that at very low energies we can ignore the s -quark (remember the idea of EFT outlined in Section 2.8). Moreover, the

up and down quark masses are quite similar between each other. The approximate flavor symmetry is reduced to $SU(2)_F$ acting on the first two components of the $SU(3)_F$ defined above (i.e. on the u and d quarks only), and is called *isospin*. More specifically we can write

$$U = \left(\begin{array}{c|c} U_2 & 0 \\ \hline 0 & 1 \end{array} \right), \quad (4.36)$$

where $U_2 \in SU(2)_F$. Notice that

$$\begin{pmatrix} u \\ d \end{pmatrix} \sim \mathbf{2} \text{ of } SU(2)_F, \quad (4.37)$$

i.e. (u, d) is an isospin doublet. Since the mass difference between the u and d quark masses is small, we expect the isospin symmetry to be a much better symmetry than $SU(3)_F$. How does $SU(2)_F$ act on the mesons? Remembering Eq. (4.24) and defining the combinations

$$\boldsymbol{\pi} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \quad (4.38)$$

the $SU(2)_F$ transformation acts as

$$\boldsymbol{\pi} \rightarrow U_2 \boldsymbol{\pi} U_2^\dagger, \quad \mathbf{K} \rightarrow U_2 \mathbf{K}, \quad \eta \rightarrow \eta. \quad (4.39)$$

We thus expect the pions to behave like a triplet of $SU(2)_F$ and the kaons to behave like a doublet. This implies that we expect the pions to have similar masses between themselves, and the kaons to have similar masses among themselves. This is precisely what happens in Nature, confirming that isospin is a pretty good symmetry. More information about the mesons' quantum numbers can be found [in these lecture notes](#). We will postpone to next section the discussion of the baryon quark composition, since a proper discussion requires the introduction of a further ingredient: color symmetry.

4.4 Color

We start by defining our notation. A generic hadronic state will be denoted by

$$\psi_{hadr} = \psi^S \times \psi^F \times \dots, \quad (4.40)$$

where ψ^S is the spin state and ψ^F is the flavor state (i.e. the quark content). The dots denote any additional quantum number that will be needed to completely define the state. It is our purpose to show how a new quantum number, the *color*, is needed for the hadron states to exist. Let us now go back to Fig. 4.3 and focus on the Δ^{++} particle. This is a spin 3/2 particle with electric charge $++$. This means that we must have $\Delta^{++} = uuu$,

with all three fermions with spin aligned in the same direction. We can write

$$\Delta^{++} = \mathcal{A}_{123} [u_{\uparrow} u_{\uparrow} u_{\uparrow}] , \quad (4.41)$$

where \mathcal{A} is the total antisymmetrization operator needed to construct the physical state out of three fermions. Clearly the antisymmetrization operation vanishes when applied over three identical states (Pauli principle), giving the absurd result $\Delta^{++} = 0$. Since the Δ^{++} particle has been observed in Nature, there is clearly some missing ingredient in our reasoning. The way out is to postulate the existence of a new quantum number, which we will call *color*. To survive the antisymmetrization of the state the color must take three different values, which we will call r (red), b (blue) and g (green). The Δ^{++} particle will be thus given by

$$\Delta^{++} = \mathcal{A}_{123} [u_{\uparrow,r} u_{\uparrow,b} u_{\uparrow,g}] , \quad (4.42)$$

where the antisymmetrization is taken over the color indices too. More in detail, we can think that any time we speak about a quark we are really speaking about three states, each one with its own color:

$$u = \begin{pmatrix} u_r \\ u_g \\ u_b \end{pmatrix} , \quad d = \begin{pmatrix} d_r \\ d_g \\ d_b \end{pmatrix} , \quad \text{etc.} \quad (4.43)$$

The compact notation we will use is u_a, d_a, \dots with the index a taking 3 possible values. An $SU(3)_c$ transformation acts only on the color indices, “shuffling” them around as $u_a \rightarrow U_a^b u_b$. Analyzing baryons decays it appears that there are no selection rules imposed by the new quantum number. This can be interpreted as an indication that the different colors of the three up quarks must combine to form a state that is a color singlet. We already know how to arrange for this: if we suppose that the color charge is associated with an $SU(3)_c$ symmetry and that the up quark transforms in the fundamental, then we immediately conclude that a color singlet can be constructed as

$$\mathbf{1} \sim \epsilon^{abc} u_{\uparrow,a} u_{\uparrow,b} u_{\uparrow,c} . \quad (4.44)$$

The same reasoning can be immediately extended to all the members of the baryon decuplet, that can be color singlets in exactly the same way:

$$B = \epsilon^{abc} q_a q_b q_c , \quad (4.45)$$

where a, b, c are color indices and each quark will have additional spin indices that we do not show explicitly. The mesons are also color singlets, since the antiquark will transform in the antifundamental of $SU(3)_c$:

$$M = \bar{q}^a q_a . \quad (4.46)$$

Let us now go back to the baryons. In Section 4.3 we did not show explicitly the quark content of the octet. As a matter of fact we discovered that two octet representations, $\mathbf{8}_{M_A}$ and $\mathbf{8}_{M_S}$, appear in the product of three fundamentals of $SU(3)_F$. Having introduced the concept of color, we are now in a position to finally discuss this point. The technical details are rather tedious and not particularly illuminating, so we will limit ourselves to point out some basic points:

- As we saw above, the correct description of a baryon state is given by

$$\psi_{hadr} = \psi^S \times \psi^F \times \psi^c, \quad (4.47)$$

where S , F and c denote, respectively, spin, flavor and color. Since baryons are fermions their state must be completely antisymmetric under quark exchange. We know from Eq. (4.45) that the color part of the state is already completely antisymmetric. To construct physical states we must thus take completely symmetric combinations $\psi^S \times \psi^F$;

- We saw in Eq. (4.30) that $\mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{1}_A + \mathbf{8}_{M_A} + \mathbf{8}_{M_S} + \mathbf{10}_S$;
- The analysis of the spin product of three quarks gives $\mathbf{2} \times \mathbf{2} \times \mathbf{2} = \mathbf{2}_{M_A} + \mathbf{2}_{M_S} + \mathbf{4}_S$;
- Putting together all this facts, we conclude that
 - The completely symmetric combination of spin 3/2 is the $\mathbf{10}_S$, since the product $\mathbf{4}_S \times \mathbf{10}_S$ is completely symmetric. This implies that spin 3/2 baryons must come in a decuplet, as observed in nature;
 - The completely symmetric combination of spin 1/2 is a linear combination of the products $\mathbf{2}_{M_A} \times \mathbf{8}_{M_A}$ and $\mathbf{2}_{M_S} \times \mathbf{8}_{M_S}$. It is interesting to observe that the quark content of both octets is the same (with different symmetry properties, as we observed above, which must be taken into account in the physical states). It is schematically given by

$$A_i^m |_{\text{quark content}} \sim B_i^m |_{\text{quark content}} \sim \begin{pmatrix} uds & uus & uud \\ dds & uds & udd \\ dss & uss & uds \end{pmatrix}. \quad (4.48)$$

The standard form of the physical baryon octet matrix is

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda^0 & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda^0 & n \\ -\Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda^0 \end{pmatrix}. \quad (4.49)$$

Following the discussion around Eq. (4.36) we conclude that, in terms of $SU(2)_F$ isospin, the Σ system behaves like a triplet while (p, n) and (Ξ^-, Ξ^0) both behave like doublets. This explains the similarity between their masses.

4.5 Additional readings

- W.K.Tung, “Group Theory in Physics”;
- H.Georgi, “Lie algebras in Particle Physics”;
- A.Bettini, “Introduction to Elementary Particle Physics”.

Chapter 5 Quantum Chromo Dynamics (QCD)

The main intuition behind QCD is the following: what happens if we associate the color charge with a force, and we identify such force with the strong interactions responsible for the existence of mesons and baryons? We will devote this chapter to the construction of such QFT. As of today, the compatibility between QCD predictions and data is remarkable, and there is consensus on the fact that QCD is the theory of strong interactions. It is nowadays included in the Standard Model of particle physics as one of the fundamental interactions of the subatomic world.

5.1 Some facts about Lie algebras

Before starting our discussion about color, it is useful to fix some notions about Lie algebras. As already pointed out in the previous section, the $SU(N)$ group is defined by the condition

$$U^\dagger U = \mathbf{1} , \quad \det U = 1 . \quad (5.1)$$

Let us start by counting the number of parameters needed to describe a $U(N)$ matrix. We know that U is a $N \times N$ complex matrix, and has thus $2N^2$ real parameters. The unitary condition $U^\dagger U = \mathbf{1}$ is a set of N^2 conditions, and we end up with $2N^2 - N^2 = N^2$ real parameters. The parameters needed to describe a $SU(N)$ matrix are instead $N^2 - 1$ (the N^2 needed to describe the unitary matrix from which we need to subtract one parameter needed to impose the $\det U = 1$ condition).

The parameters needed to describe the $SU(N)$ matrices take values in \mathbb{R} and are among the groups called *Lie groups*. The theory of Lie groups is a very reach subject, and we will not enter in any detail here. It is sufficient to remind the reader the following fundamental fact: any element of the Lie group can be reconstructed studying the behavior of a neighborhood of the identity, where we can always write

$$U = e^{i\alpha^A T^A} . \quad (5.2)$$

where the α^A 's are $N^2 - 1$ small real parameters. The $N \times N$ matrices T^A are called the *generators of the Lie algebra*. They are hermitian matrices. This can be easily shown expanding the exponential form of the group element up to first order in the parameters.

We obtain

$$U^\dagger U \simeq (\mathbf{1} - i\alpha^A (T^A)^\dagger) (\mathbf{1} + i\alpha^A T^A) \simeq \mathbf{1} + i \underbrace{(T^A - (T^A)^\dagger)}_0. \quad (5.3)$$

The $\det U = 1$ condition can be implemented remembering the identity

$$\det(e^A) = e^{\text{tr}A}, \quad (5.4)$$

from which we conclude that the generators must be traceless, $\text{tr}T^A = 0$. An interesting property emerges when considering the group closure property $U_1 U_2 = U_3$ stating that the product of two group elements is a group element. Writing them in exponential form, we need to guarantee that

$$e^{i\alpha^A T^A} e^{i\beta^A T^A} = e^{i\gamma^A T^A} \quad (5.5)$$

for some set of parameters γ^A . Expanding up to second order in the parameters we can show that the only way for the previous equation to be true is if the commutator $[T^A, T^B] = T^A T^B - T^B T^A$ is proportional to a generator,

$$[T^A, T^B] = i f^{ABC} T^C. \quad (5.6)$$

The real numbers f^{ABC} are called “structure constants” of the algebra. It can be shown that the structure constants are completely antisymmetric in their indices, and that the Lie algebra obeys the Jacobi identity


$$[T^A, [T^B, T^C]] + [T^B, [T^C, T^A]] + [T^C, [T^A, T^B]] = 0, \quad (5.7)$$

which can be immediately translated in terms of the structure constants:

$$f^{BCM} f^{AMN} + f^{CAM} f^{BMN} + f^{ABM} f^{CMN} = 0. \quad (5.8)$$

This has an interesting consequence: rearranging the indices and defining $-i f^{ABC} = (t^A)_{BC}$ we obtain

$$[t^A, t^B]_{CN} = i f^{ABM} (t^M)_{CN}. \quad (5.9)$$

 **Exercise 5.1** Check explicitly that the previous formula is true.

This means that the structure constants fill up the matrix elements of a representation of the group. Since the indices C and N take values between 1 and $N^2 - 1$ (because they must equal the number of generators), this representation has dimension $N^2 - 1$, exactly the dimension of the adjoint representation we found in the discussion around Eq. (4.22). To make contact with Eq. (4.24) we observe that *any* complex $N \times N$ matrix M can always be decomposed as $M = M^0 \mathbf{1} + M^A T^A$, where $\{M^0, M^A\}$ are a set of

complex parameters. If this matrix transforms in the adjoint then it must be traceless, and we end up with $M = M^A T^A$. Its transformation is given by

$$\begin{aligned}
 M &\rightarrow U M U^\dagger = (\mathbf{1} + i\alpha^A T^A) M (\mathbf{1} - i\alpha^A T^A) \\
 &= M + i\alpha^A [T^A, M] \\
 &= M^C T^C + i\alpha^A M^B [T^A, T^B] \\
 &= [M^C - f^{ABC} \alpha^A M^B] T^C
 \end{aligned} \tag{5.10}$$

$$\Rightarrow M^C \rightarrow M^C - f^{ABC} \alpha^A M^B = M^C + i\alpha^A (t^A)_{CB} M^B,$$

which is exactly what we expect from an object transforming in the adjoint representation.

Let us conclude with the explicit form of the generators of the $SU(2)$ and $SU(3)$ groups. For the former group we have

$$T^A = \frac{\sigma^A}{2}, \tag{5.11}$$

where σ^A are the Pauli matrices, while for $SU(3)$ we have

$$T^A = \frac{\lambda^A}{2}, \tag{5.12}$$

where λ^A are the so-called Gell-Mann matrices

$$\begin{aligned}
 \lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
 \lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
 \lambda^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
 \end{aligned} \tag{5.13}$$

We use the standard nomenclature for these matrices. For both $SU(2)$ and $SU(3)$ we are normalizing the generators as

$$\text{tr}(T^A T^B) = \frac{\delta^{AB}}{2}. \tag{5.14}$$

5.2 Color transformations and gluons

The first step we need to take is to understand how to construct a theory in which the charge associate itself in triplets of $SU(3)_c$. This means that any of the quarks

introduced so far (u , d and s) will transform as

$$q_i \rightarrow U_i^j q_j, \quad (5.15)$$

where $U \in SU(3)_c$. In order to construct the theory of strong interactions we proceed as we did for QED, coupling the new mediators (called gluons) to some conserved current associated with $SU(3)_c$. To compute the charge associated with such symmetry we start from the free quark Lagrangian

$$\mathcal{L}_{QCD} = \bar{q} (i\cancel{\partial} - m) q, \quad (5.16)$$

and we apply Noether theorem. According to the discussion in Section 2.2 we need to consider the infinitesimal transformation

$$U \simeq \mathbf{1} + i\alpha^A T^A, \quad (5.17)$$

where the T^A are the generators of the group belonging to the Lie algebra. The Noether current associated with this transformation is

$$J^{A\mu} = \bar{q} \gamma^\mu T^A q, \quad (5.18)$$

thus we see that we have one conserved current for each generator. As we saw in Sec 5.1 and $SU(N)$ group has $N^2 - 1$ generators. It is important to notice that, in general, the generator will have off-diagonal elements, in such a way that the current connects different types of fermions. Let us now introduce the gluons. Since we have $N^2 - 1$ generators, we need to introduce the same number of gluons, and couple each one to one current. We obtain

$$\begin{aligned} \mathcal{L}_{QCD} &= \bar{q} (i\cancel{\partial} - m) q - g_s G_\mu^A \bar{q} \gamma^\mu T^A q \\ &= i\bar{q} \gamma^\mu (\partial_\mu + ig_s G_\mu^A T^A) q - m\bar{q}q. \end{aligned} \quad (5.19)$$

Once more we see that we can write the interactions in our theory in terms of a covariant derivative. The parameter g_s appearing there is strong coupling (or QCD coupling). What about the gluon's kinetic term? Things are more complicated with respect to the QED case. To appreciate the difference let us first study how the gluons transform under an $SU(3)_c$ transformation. We can obtain this information requiring $D_\mu q \rightarrow U D_\mu q$, i.e. requiring the covariant derivative to transform covariantly. Defining the matrix $G_\mu \equiv G_\mu^A T^A$ we obtain

$$\begin{aligned} (D_\mu q)' &= (\partial_\mu + ig_s G'_\mu) q' \\ &= \partial_\mu (Uq) + ig_s G'_\mu Uq \\ &= (\partial_\mu U)q + U\partial_\mu q + ig_s G'_\mu Uq \\ &\stackrel{!}{=} U\partial_\mu q + ig_s UG_\mu q. \end{aligned} \quad (5.20)$$

Solving for G'_μ we obtain

$$G'_\mu = U G_\mu U^\dagger + \frac{i}{g_s} (\partial_\mu U) U^\dagger . \quad (5.21)$$

It is interesting to compute the transformation of each of the G_μ^A components, expanding the expression above in the Lie algebra basis. We obtain

$$\begin{aligned} (G_\mu^A)' T^A &= (\mathbf{1} + i\alpha^B T^B) G_\mu^C T^C (\mathbf{1} - i\alpha^D T^D) + \frac{i}{g_s} (i\partial_\mu \alpha^A) T^A (\mathbf{1} - i\alpha^B T^B) \\ &= G_\mu^A T^A + i\alpha^B G_\mu^C [T^B, T^C] - \frac{1}{g_s} \partial_\mu \alpha^A T^A . \end{aligned} \quad (5.22)$$

The expression above can be written in terms of the structure constants $[T^B, T^C] = if^{BCA} T^A$ as

$$G_\mu^A \rightarrow G_\mu^A - \frac{1}{g_s} \partial_\mu \alpha^A - f^{ABC} \alpha^B G_\mu^C . \quad (5.23)$$

We stress two interesting facts: (i) the second term on the right hand side of Eq. (5.23) corresponds to the usual gauge transformation, as expected for a massless particle, and (ii) the third term is a non-trivial transformation appearing due to the non-abelian nature of the $SU(3)_c$ group. This behavior is different from the one we discussed for QED, and is reminiscent of what happens for matter fields, which have non-trivial transformations under the symmetry group because of their non-vanishing charge. The conclusion is thus immediate: unlike the photon, the eight gluons carry color charge and, as a consequence, they behave non-trivially under a symmetry transformation. Nevertheless, comparing with the transformation of fields in the fundamental, Eq. (5.17), we see that the gluons do not transform in the fundamental but rather in the adjoint.

5.3 The gluon kinetic term and the complete QCD Lagrangian

We now discuss how to write down the gluon kinetic term. We know that the correct equations of motion for a massless spin-1 field are obtained if a term $-(\partial_\mu G_\nu^A - \partial_\nu G_\mu^A)^2/4$ is present in the Lagrangian. We immediately identify a problem: this kinetic term is invariant under the gauge transformation alone, but is not invariant once the non-abelian term in Eq. (5.23) is considered. To solve the problem we will try to add some terms to recover gauge invariance (as we did in Section 3.3). Let us first compute the

transformations of $\partial_\mu G_\nu - \partial_\nu G_\mu$:

$$\begin{aligned}
 \partial_\mu G_\nu - \partial_\nu G_\mu &\rightarrow \partial_\mu \left(U G_\nu U^\dagger + \frac{i}{g_s} \partial_\nu U U^\dagger \right) - \partial_\nu \left(U G_\mu U^\dagger + \frac{i}{g_s} \partial_\mu U U^\dagger \right) \\
 &= U (\partial_\mu G_\nu - \partial_\nu G_\mu) U^\dagger + [(\partial_\mu U) G_\nu - (\partial_\nu U) G_\mu] U^\dagger \\
 &\quad - U [G_\mu (\partial_\mu U^\dagger) - G_\nu (\partial_\nu U^\dagger)] + \frac{i}{g_s} [(\partial_\nu U) (\partial_\mu U^\dagger) - (\partial_\mu U) (\partial_\nu U^\dagger)] .
 \end{aligned} \tag{5.24}$$

We see that only the first term has a simple transformation under the action of the symmetry. In order to cancel the other terms we try with the simplest object that can be constructed out of G_μ and is antisymmetric under $\mu \leftrightarrow \nu$ exchange, the commutator $[G_\mu, G_\nu]$:

$$\begin{aligned}
 [G_\mu, G_\nu] &\rightarrow \left[U G_\mu U^\dagger + \frac{i}{g_s} \partial_\mu U U^\dagger, U G_\nu U^\dagger + \frac{i}{g_s} \partial_\nu U U^\dagger \right] \\
 &= U [G_\mu, G_\nu] U^\dagger - \frac{i}{g_s} [(\partial_\mu U) G_\nu - (\partial_\nu U) G_\mu] U^\dagger \\
 &\quad - \frac{i}{g_s} U [G_\mu (\partial_\nu U^\dagger) - G_\nu (\partial_\mu U^\dagger)] - \frac{1}{g_s^2} [(\partial_\nu U) (\partial_\mu U^\dagger) - (\partial_\mu U) (\partial_\nu U^\dagger)]
 \end{aligned} \tag{5.25}$$

where we have used the property $(\partial_\mu U) U^\dagger = -U (\partial_\mu U^\dagger)$.

Exercise 5.2 Prove that the identity $(\partial_\mu U) U^\dagger = -U (\partial_\mu U^\dagger)$ is true.

Comparing the two expressions we see that the combination

$$G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu + i g_s [G_\mu, G_\nu], \Rightarrow G_{\mu\nu}^A = \partial_\mu G_\nu^A - \partial_\nu G_\mu^A - g_s f^{BCA} G_\mu^B G_\nu^C \tag{5.26}$$

transforms in the adjoint and contains the usual field strength. An invariant term in the Lagrangian is thus given by

$$\mathcal{L}_{\text{gluon}} = \kappa \text{tr}[(G_{\mu\nu})^2], \tag{5.27}$$

where the constant κ must be chosen to guarantee canonical kinetic terms. Using the normalization of Eq. (5.14) for the generators the previous Lagrangian amounts to:

$$\mathcal{L}_{\text{gluon}} = \kappa \text{tr}[T^A T^B] G_{\mu\nu}^A G_{\mu\nu}^B = \frac{\kappa}{2} \delta^{AB} G_{\mu\nu}^A G_{\mu\nu}^B, \tag{5.28}$$

from which we conclude that the correct choice is $\kappa = -1/2$. Finally, the gluon kinetic term is given by

$$\mathcal{L}_{\text{gluon}} = -\frac{1}{4} (G_{\mu\nu}^A)^2, \tag{5.29}$$

with $G_{\mu\nu}^A$ defined in Eq. (5.26).

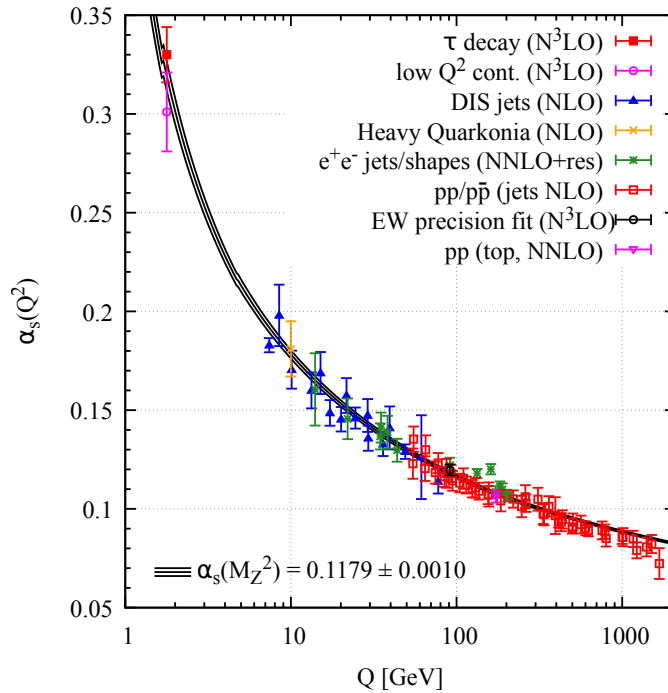


Figure 5.1: Variation of the strong coupling with energy: points are experimental data, while the continuous line is the QCD prediction. Taken from this link [this link](#).

The complete QCD Lagrangian is given by

$$\mathcal{L}_{QCD} = -\frac{1}{4}(G_{\mu\nu}^A)^2 + \bar{q}(i\not{D} - m_q)q. \quad (5.30)$$

This Lagrangian is essentially the classical Lagrangian for QCD, and it is valid in the tree approximation (i.e. when no loops are considered) once a gauge fixing term is introduced. The extension to higher orders in perturbation theory requires however one more ingredient, the so-called *ghost* Lagrangian. The problem is once more connected to the longitudinal degrees of freedom present in the gluon field: at loop level, these states contribute to the physical amplitude, and spoil completely gauge invariance. The invariance can be recovered adding new unphysical fields (the ghosts, which being unphysical appear only in the internal lines of any amplitude) with precisely the coefficient needed to cancel the contribution from the unphysical gluon polarization. Although the procedure seems rather ad-hoc the way we introduced it, it is justified by the Faddeev-Popov method (see any advanced book on QFT for a thorough treatment, and Aitchison, ch. 4, for a more intuitive treatment).

5.4 Radiative corrections

As we saw in Sec. 3.5 in the case of QED, radiative corrections induce a running of the coupling. This is a general result in QFT, and QCD is no exception. It can be shown

that the β function of the coupling of a non-abelian gauge theory is

$$\frac{dg}{d \log \mu} = \beta_g = -\frac{g^3}{16\pi^2} \left(\frac{11}{3} T(R_{Ad}) - \frac{2}{3} T(R_F) - \frac{1}{3} T(R_S) \right). \quad (5.31)$$

The quantity $T(R_i)$ is the *index* of the representation R_i , for the gauge fields (adjoint), Weyl fermion fields F and complex scalar fields S , defined in the terms of the generators T_R^A as

$$\text{tr}(T_R^A T_R^B) = T(R) \delta^{AB}. \quad (5.32)$$

The trace is taken over all species. For a non-abelian $SU(N)$ theory, explicit computation gives

$$T(R_{Ad}) = N, \quad T(R_{\text{fund}}) = \frac{1}{2} \quad (5.33)$$

for the adjoint and fundamental representations, respectively. In the case of QED we recover the result obtained in Sec. 3.5 once we consider that both the LH and RH fermion must be considered, and we have

$$T(R_F) = \sum_{f_L} Q_{f_L}^2 + \sum_{f_R} Q_{f_R}^2 = 2 \sum_f Q_f^2 \Rightarrow \beta_e = \left(\sum_f Q_f^2 \right) \frac{e^3}{12\pi^2}. \quad (5.34)$$

We stress that $\beta_e > 0$ independently on the number of fermions. Let us now consider the case of QCD:

$$\begin{aligned} \beta_{g_s} &= -\frac{g_s^3}{16\pi^2} \left(\frac{11}{3} \times 3 - \frac{2}{3} \times 2n_q \right) \\ &= -\frac{g_s^3}{16\pi^2} \frac{33 - 2n_q}{3}, \end{aligned} \quad (5.35)$$

where n_q denotes the number of quarks contributing to the running. Since we will have at most $n_q = 6$, we conclude that, unlike QED, $\beta_{g_s} < 0$, i.e. the QCD coupling grows larger and larger at low energies. Gauge theories in which this happens are called *asymptotically free*, since their coupling becomes smaller at higher energies, in such a way that they approximate a free theory in this regime. This result was derived in 1973 by Politzer and, independently, by Gross and Wilczek, and it awarded the trio the Nobel prize in 2004. As we are going to see in Section 6, the fact that the QCD coupling becomes strongly interacting at low energy is believed to be the origin of the meson and baryon spectra discussed at the beginning of this chapter.

We conclude this section noticing that a useful measure of perturbativity/non-perturbativity can be given in terms of

$$\alpha_s \equiv \frac{g_s^2}{4\pi}. \quad (5.36)$$

More specifically, we consider the theory non-perturbative when $\alpha_s \ll 1$, while the limit

of perturbativity is reached when $\alpha_s \sim 1$. For QCD this happens around a scale

$$\Lambda_{QCD} \sim 1 \text{ GeV} . \quad (5.37)$$

We present in Fig. 5.1 the variation of the strong coupling with energy.

5.5 Extracting α_s from experiments

The strong coupling α_s can be measured in several physical processes. Clearly we cannot measure the value of α_s in the very low energy regime, since in this energy range the predictions of perturbation theory are not reliable. Measurements can however be performed for higher energy. We show in Fig. 5.2 a summary of measurements with their errors. We can see that, with few exceptions, all the data samples are compatible with the others at the 1σ level.

5.6 Additional readings

- S.Weinberg, “The Quantum Theory of Fields” vol. 1;
- M.Peskin and D.Schroeder, “An Introduction To Quantum Field Theory”;
- S.Coleman, “Lectures on Quantum Field Theory”;
- M.Schwartz, “Quantum Field Theory and the Standard Model”;
- D. B. Kaplan, “Lectures on Effective Field Theory” ([link](#));
- R. Penco, “An Introduction to Effective Field Theories” ([link](#));

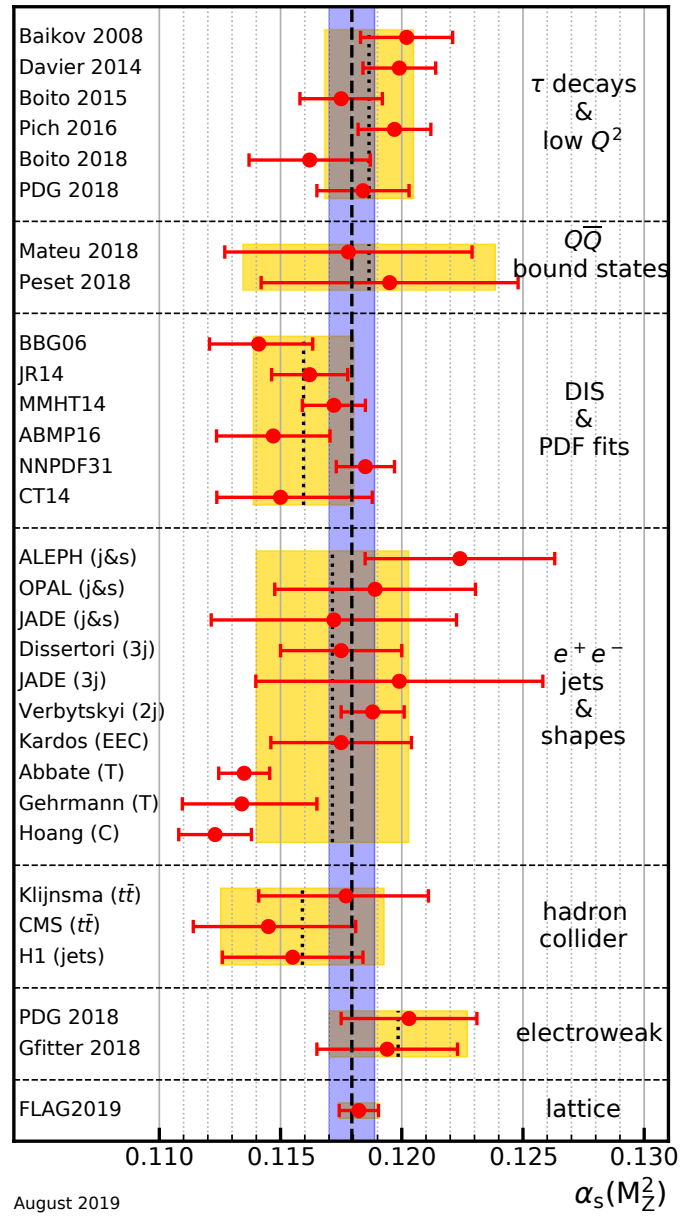


Figure 5.2: Comparison between different measurements of the strong coupling. Taken from [this link](#).

Chapter 6 Confinement, the emergence of mesons and spontaneous symmetry breaking

In this section we will discuss qualitatively the emergence of the meson and baryons bound states in the regime in which the QCD coupling becomes non-perturbative. Since in this regime our perturbative techniques cannot be used, the QCD Lagrangian is useless to make predictions. We will then turn to the problem of how to find appropriate effective Lagrangians apt to describe the essential features of the physics of mesons.

6.1 Confinement and its heuristic consequences

Let us start by commenting that there is still no formal proof of confinement in any non-abelian gauge theory. ¹ We can nevertheless outline the qualitative picture which has been established over the last decades. As usual, let us start with a QED analogy. Take two electrically charged particles at a certain distance d . They interact via photon exchange, which are not self-interacting. As d is increased, the electric field lines tend to spread out, diminishing the electric flux in the region between the two charges. Consider now QCD in the non-perturbative regime, and consider the force between two quarks. It is mediated by gluon exchange. As the distance between the two quarks increases the chromo-electric field does not spread out because of self-interactions, but tends to form a “flux tube” (or string) in the region between the two quarks, effectively binding the quarks in a bound state. Increasing the distance between the two quarks the energy per unit length of the string increases linearly with the distance, until it becomes energetically favorable to create a quark anti-quark pair out of the vacuum and two new strings are formed. ²

It is believed that an effective way to describe color confinement is to state that the expectation value of the quark bilinear in the vacuum is driven to a non-vanishing value by the non-perturbative interactions, i.e.

$$\langle 0 | \bar{q}q | 0 \rangle \neq 0 . \quad (6.1)$$

¹The problem of confinement is in fact one of the seven Millennium Prize Problems, whose solution would award USD 1 million.

²Historically, one of the first motivations for the study of string theory came from the study of confinement and hadron physics.

Notice that the operator $\bar{q}q$ is a Lorentz scalar. As we are going to see in next section, when scalar operators have a non-vanishing expectation value with the vacuum the phenomenon of *spontaneous symmetry breaking* may occur. The expectation value of any operator in the vacuum of the theory is called Vacuum Expectation Value (vev).

6.2 Spontaneous symmetry breaking

As we discussed above, in the non-perturbative regime of QCD our perturbative computations are completely unreliable. It would thus seem that the description of mesons and baryons is an impossible task. As we are now going to see this is not so, and as a matter of fact it is possible to write an effective Lagrangian that describe the mesons (and the baryons, in some energy regime). The starting point is Eq. (6.1) and the QCD Lagrangian, Eq. (5.30). The masses of the light quarks u , d and s are much lighter than the QCD scale in Eq. (5.37) (we remind that $m_u \sim m_d \sim \text{MeV}$, while $m_s \sim 100 \text{ MeV}$). When considering QCD in its non-perturbative regime we can thus, in first approximation, neglect the quark masses.³ With this approximation the QCD Lagrangian is given by

$$\mathcal{L}_{QCD} = -\frac{1}{4}(G_{\mu\nu}^A)^2 + \bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R, \quad (6.2)$$

where we remind the reader that $q = (u, d, s)^T$. We have explicitly separated the LH from the RH spinor components to stress that each quark kinetic term is invariant under the transformations

$$q_L \rightarrow U_L q_L, \quad q_R \rightarrow U_R q_R, \quad (6.3)$$

where $U_{L,R}$ are two different unitary matrices. The classical QCD Lagrangian in the massless quark limit is thus invariant under $U(3)_L \times U(3)_R$ transformations. This classical symmetry that acts in a different way on the different chiral components of the quark field is called *chiral symmetry*. It is convenient to decompose the symmetry group as

$$U(3)_L \times U(3)_R = U(1)_V \times U(1)_A \times SU(3)_L \times SU(3)_R. \quad (6.4)$$

The $U(1)_V$ symmetry is called the “vector” $U(1)$ because the LH and RH components transform with the same phase. The corresponding charge is called “baryon number”. It remains an exact symmetry even when the quark masses are turned on again. The

³The heavier quarks have all masses above Λ_{QCD} and will not play any role in what follows. The c and b quarks can form meson states, but they cannot be described with the techniques to be developed in this section. The t quark, on the other hand, is very short lived and it is believed to decay much before any bound state can be formed.

$U(1)_A$ symmetry is called “axial” because the LH and RH components transform with opposite phases. Although we will not enter in details, this is an example of anomalous symmetry. The remaining chiral symmetry $SU(3)_L \times SU(3)_R$ is non-anomalous, but only an approximate symmetry of QCD, since it is explicitly broken by the quark masses.

Exercise 6.1 Show explicitly that the quark mass term is not invariant under a chiral transformation, making thus the chiral symmetry only an approximate rather than an exact symmetry.

Let us now see what happens to the classical symmetries when we transform the vacuum in Eq. (6.1). First, let us write the vacuum as

$$\langle 0 | \bar{q}q | 0 \rangle = \langle 0 | \bar{q}_R q_L | 0 \rangle + h.c. \neq 0 \quad (6.5)$$

Under the non-anomalous symmetries we obtain

$$\begin{aligned} U(1)_V : \langle 0 | \bar{q}_R q_L | 0 \rangle &\rightarrow \langle 0 | \bar{q}_R e^{-i\alpha_B} e^{i\alpha_B} q_L | 0 \rangle = \langle 0 | \bar{q}_R q_L | 0 \rangle, \\ SU(3)_L \times SU(3)_R : \langle 0 | \bar{q}_R q_L | 0 \rangle &\rightarrow \langle 0 | \bar{q}_R U_R^\dagger U_L q_L | 0 \rangle. \end{aligned} \quad (6.6)$$

We see that the vacuum is invariant under a baryon number transformation, but is invariant under a chiral transformation only for the special case $U_L = U_R$. We are thus encountering a situation in which *a symmetry of the Lagrangian is not a symmetry of the vacuum*. Whenever this happens we say that the symmetry is *spontaneously broken*. The remaining symmetry of the vacuum is called “unbroken group”. In the case of chiral symmetry, the breaking pattern is

$$SU(3)_L \times SU(3)_R \rightarrow SU(3)_V, \quad (6.7)$$

where $SU(3)_V$ is the vector subgroup for which LH and RH components transform in the same way, $U_L = U_R$.

The physics of spontaneous symmetry breaking is very reach, and we defer the reader to QFT texts for thorough treatments. Here we instead take a semi-heuristic approach to get to some useful results.

6.2.1 Spontaneous symmetry breaking for a perturbative system

As we have seen in the previous section, the vacuum expectation value (vev) of the $\bar{q}q$ operator dictates the symmetries of the theory. To be more precise, let us go back to Eq. (2.16), i.e. to the transformation of a field under a symmetry transformation:

$$\delta\phi_0(x) = i [Q, \phi_0(x)] \equiv \phi(x).$$

In the last step we have simply recognized that the variation $\delta\phi_0(x)$ of the $\phi_0(x)$ field must itself be a field, that we will call $\phi(x)$. We now take the expectation value of $\phi(0)$ between two vacuum states:

$$\langle 0 | \phi(x) | 0 \rangle \equiv \langle \phi \rangle = i \langle 0 | Q \phi_0(x) - \phi_0(x) Q | 0 \rangle. \quad (6.8)$$

We have two possible situations:

1. $\langle \phi \rangle = 0$ is consistent only if $Q | 0 \rangle = 0$, i.e. if the vacuum is invariant under the symmetry transformation. In this case we say that the symmetry is conserved, or that it is realized in the Wigner-Weyl mode;
2. $\langle \phi \rangle \neq 0$ is consistent only if $Q | 0 \rangle \neq 0$, i.e. when the vacuum is not invariant under the symmetry transformation. In this case we say that the symmetry is spontaneously broken, or that it is realized in the Nambu-Goldstone mode.

The second situation correspond to the results we have seen in the previous section, in which the vev of an operator dictates the symmetry of the theory.

It is important to stress an important consequence of this discussion: only scalar operators can acquire a non-vanishing vev. To understand this point let us suppose that $\langle \psi \rangle \neq 0$ for a fermion field (the same reasoning can be repeated for fields with any spin > 0). Since under a Lorentz transformation we have $\psi \rightarrow S\psi$, we immediately see that $S\langle \psi \rangle \neq 0$, i.e. that Lorentz symmetry is spontaneously broken. Physically this means that it is no longer true that all the inertial reference frames are equivalent, in stark contrast with experimental results ⁴. The only way to break internal symmetries without breaking Lorentz symmetry at the same time is to allow only scalar operators to acquire non-trivial vevs. This is precisely what happens with the $\bar{q}q$ operator: although it is made by quark fields, the combination $\bar{q}q$ is a Lorentz scalar, and as such can acquire a non-vanishing vev without breaking Lorentz symmetry.

Having established that to determine whether spontaneous symmetry breaking (SSB) is happening we need to study the vev of scalar operators, we now need to understand how to compute such vevs. Let us first start from a completely classical field theory. In this case the vacuum is identified by the configuration of the ϕ field that corresponds to minimum energy, i.e. that constant field value (to be called v) that minimizes the potential. Notice that v is a solution of the equation of motion $\mathcal{L}'[v] = 0$ and is thus a stationary configuration of the action. To translate such result to the quantum theory we can use the path integral approach to QFT, since it allows to compute $\langle \phi \rangle$. In the semiclassical limit $\hbar \rightarrow 0$, the path integral will be dominated by the stationary

⁴The possibility of Lorentz symmetry violation has been extensively searched for experimentally. Up to now no experimental deviation from the predictions of Lorentz symmetry has been found.

configuration v and the ϕ expectation value results

$$\langle 0|\phi|0\rangle = \lim_{\hbar \rightarrow 0} \int \mathcal{D}\phi \phi e^{\frac{i}{\hbar}S(\phi)} = v. \quad (6.9)$$

We can thus have a semiclassical idea of whether a theory breaks spontaneously a symmetry analyzing its potential.⁵

Consider now for simplicity the simplest possible case: a scalar theory with a $U(1)$ symmetry. The potential in this case is given by

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 = \frac{1}{2} \mu^2 (a^2 + b^2) + \frac{\lambda}{4} (a^2 + b^2)^2, \quad (6.10)$$

where we have written the complex ϕ field in terms of its real and imaginary part as usual, $\phi = (a + ib)/\sqrt{2}$. This potential is extremized by (from now on we will use the short handed notation $\langle \phi^\dagger \phi \rangle$ to indicate the vev of some operator, $\phi^\dagger \phi$ in this case)

$$v^2 = \langle \phi^\dagger \phi \rangle = \langle a \rangle^2 + \langle b \rangle^2 = \begin{cases} 0 & \text{for } \mu^2 > 0 \\ \sqrt{-\frac{\mu^2}{\lambda}} & \text{for } \mu^2 < 0 \end{cases} \quad (6.11)$$

Remembering that the parameters in the Lagrangian (and hence in the potential) are not physical parameters, we can admit a negative μ^2 . We see that the extremum equation identifies a family of vacua $\langle \phi^\dagger \phi \rangle$ without specifying any phase, making all these vacua equivalent. Let us now study when the extrema can be a minimum. To this end, we need to analyze the matrix of second derivatives of the potential, computed in the extremum. Using the $\mathbf{f} = (a, b)$ basis we obtain

$$V''|_{\text{extremum}} = \begin{pmatrix} \mu^2 + \lambda(3\langle a \rangle^2 + \langle b \rangle^2) & 2\lambda\langle a \rangle\langle b \rangle \\ 2\lambda\langle a \rangle\langle b \rangle & \mu^2 + \lambda(\langle a \rangle^2 + 3\langle b \rangle^2) \end{pmatrix}. \quad (6.12)$$

When $\mu^2 > 0$ we have $\langle a \rangle = 0 = \langle b \rangle$, and we see that the two scalars are degenerate, both with squared mass equal to μ^2 . When $\mu^2 < 0$ we can parametrize the vacua as $\langle a \rangle = v \cos \alpha$, $\langle b \rangle = v \sin \alpha$, and the matrix of second derivatives becomes

$$V''|_v = -2\mu^2 \begin{pmatrix} \cos^2 \alpha & \sin \alpha \cos \alpha \\ \sin \alpha \cos \alpha & \cos^2 \alpha \end{pmatrix}. \quad (6.13)$$

This matrix has vanishing determinant, signaling the existence of a massless physical state. Such state is called a Nambu-Goldstone boson (NGB). As we are now going to show, the appearance of NGBs is typical of spontaneous symmetry breaking. However, before turning to the general proof of the NGBs masslessness, let us observe that computing the EoM from the potential above gives

$$\begin{aligned} (\square + \mu^2)a &= -\lambda a(a^2 + b^2) \\ (\square + \mu^2)b &= -\lambda b(a^2 + b^2) \end{aligned} \quad (6.14)$$

⁵Quantum loops can be systematically included in this approach via the so-called Coleman-Weinberg 1-loop potential.

which are clearly incorrect, since they predict a negative squared scalar mass. The problem is that we are expanding the theory around the $\langle\phi\rangle = 0$ vacuum, instead that around the correct non-trivial vacuum. To correct for this inconsistency we shift the scalar fields according to

$$\phi \rightarrow \langle a \rangle + i\langle b \rangle + \frac{a + ib}{\sqrt{2}} = ve^{i\alpha} + \frac{a + ib}{\sqrt{2}}, \quad (6.15)$$


where the new a and b fields have vanishing vevs. Moreover, we already know that one scalar state should completely disappear from the potential to ensure that its mass is always vanishing. This can be achieved explicitly writing the scalar field in the parametrization

$$\phi = e^{\frac{i\pi(x)}{\sqrt{2}v}} \left(v + \frac{h(x)}{\sqrt{2}} \right). \quad (6.16)$$

In this parametrization $\pi(x)$ is the NGB field, while $h(x)$ is the massive field. We choose the exponential parametrization for the NGB because it automatically guarantees its disappearance from the potential, since V depends only on the combination $\phi^\dagger\phi$. It will be important in the following to notice that the exponential parametrization makes clear that the NGB shifts under a global $U(1)$,

$$\phi \rightarrow e^{i\alpha}\phi \quad \Rightarrow \quad \pi(x) \rightarrow \pi(x) + \sqrt{2}v\alpha. \quad (6.17)$$

This shift symmetry explains why no potential can be present for the NGB, since all $\pi(x)$ polynomials necessarily break the symmetry. The only terms which are allowed are those that involve $\partial_\mu\pi(x)$, which are clearly invariant under the shift symmetry since α is a constant.

 **Exercise 6.2** Verify explicitly that the NGB disappears from the potential (but not from the kinetic term). Compute explicitly the mass of the h scalar.

6.2.2 Goldstone theorem

We will now generalize the results of our $U(1)$ example to more general situations.

Consider a theory of n real scalar fields $\phi_i, i = 1, \dots, n$ which is invariant under a continuous symmetry G . Suppose moreover that there is a vacuum configuration $\mathbf{v} = (v_1, \dots, v_n)$ which is left invariant by a subgroup $H \subset G$, in such a way that

$$h\mathbf{v} = \mathbf{v}, \quad h \in H. \quad (6.18)$$

The mass matrix $\partial^2 V / \partial\phi_i \partial\phi_j$ has $\dim G - \dim H$ vanishing eigenvalues, correspond-

ing to the NGBs of the theory.

To prove the result we observe that the invariance of the theory under a G transformation can be stated as

$$V(\phi) = V(\phi + i\alpha^A T^A \phi) \simeq V(\phi) + i\alpha^A T_{ij}^A \phi_j \frac{\partial V}{\partial \phi_i}. \quad (6.19)$$

This implies that

$$T_{ij}^A \phi_j \frac{\partial V}{\partial \phi_i} = 0 \quad \forall A. \quad (6.20)$$

We now compute the mass matrix deriving the previous equation:

$$\frac{\partial^2 V}{\partial \phi_m \partial \phi_i} T_{ij}^A \phi_j + \frac{\partial V}{\partial \phi_i} T_{ij}^A \delta_{mj} = 0 \xrightarrow{\text{minimum}} \mathcal{M}_{mi}^2 T_{ij}^A v_j = 0 \quad (6.21)$$

We see that there are two possibilities:

1. the generator of H are defined as those generators that annihilate the vacuum. For these operators the previous equation is automatically satisfied;
2. the G generators that are not in H do not annihilate the vacuum. This implies that $T_{ij}^A v_j \neq 0$ must correspond to a vanishing eigenvalue of the scalar mass matrix.

The number of vanishing eigenvalues equals the number of generators for which $T_{ij}^A v_j \neq 0$, which is $\dim G - \dim H$. The generators that do not annihilate the vacuum generate the *coset* G/H .

There is a nice way to write the scalar field that simplifies the derivation of the physical properties of the NGBs. As we saw above, the NGBs are associated with the directions $T_{ij}^A v_j \neq 0$. Moreover, we know that their mass must vanish, implying that they must somehow disappear from the scalar potential (for if they had a non-trivial potential, they would in general have a non-vanishing mass). Let us start from the vacuum configuration \mathbf{v} . To simplify the notation, let us call U^A the unbroken generators (the generators of H) and B^A the broken generators (the generators of G/H). We take them to satisfy the orthogonality condition $\text{tr}(B^A U^B) = 0$. Moreover, the commutation relations are in general

$$[U^A, U^B] = i f^{ABC} U^C, \quad [U^A, B^\alpha] = i f^{A\alpha\beta} B^\beta, \quad [B^\alpha, B^\beta] = i f^{\alpha\beta\gamma} B^\gamma + i f^{\alpha\beta A} U^A, \quad (6.22)$$

where, for clarity of notation, we have denoted with capital latin letters the indices of the unbroken generators and with greek letters the indices of the broken generators.

To the vacuum configuration we add the scalar fields which are not NGBs, i.e. those associated with the eigenvectors $U^A \mathbf{v} = 0$. We call \mathbf{h} such scalars. Since they are by definition the scalar associated with the unbroken generators, it is clear that the \mathbf{h} vector is parallel to \mathbf{v} in the scalar field space. We now apply a G transformation to this state,

obtaining

$$\phi = e^{i\frac{\pi^A(x)B^A}{\sqrt{2}v}} \left(\mathbf{v} + \frac{\mathbf{h}}{\sqrt{2}} \right). \quad (6.23)$$

In writing the previous expression we have inserted a factor $1/\sqrt{2}$ to properly normalize the real scalar fields, and we have used the fact that only the B^A generators give a non-vanishing action on the vev. In addition, we have promoted the parameters of the G/H transformation to NGBs fields, normalized to $v = \|\mathbf{v}\|$. Notice that since by definition any G transformation leaves the potential invariant, as we saw in the $U(1)$ case above the vacuum is not unique and we can always apply any G transformation to align along any convenient direction. The physics will not depend on our choice.

6.3 Chiral symmetry and the CCWZ construction

After developing some essential facts about spontaneous symmetry breaking, we are in the position to go back to chiral symmetry breaking. As we saw in our discussion around Eq. (6.6), the quark condensate plays the role of the vacuum \mathbf{v} of the previous discussion, and drives the breaking

$$SU(3)_L \times SU(3)_R \rightarrow SU(3)_V. \quad (6.24)$$

The main point that will allow us to write a Lagrangian describing the light mesons is the following: since all the mesons composed by the u , d and s quark are much lighter than the QCD scale Λ_{QCD} , we postulate that they are NGBs of the spontaneous breaking of chiral symmetry. In a first stage we will set to zero the quark masses, and work with massless mesons. We will later on see how to generate the small mesons masses from the small quark masses. Let us first count the number of states to confirm that there is a hope that our treatment gives the correct Lagrangian. If everything works, we should obtain an octet of mesons. This is indeed the case: according to Goldstone theorem, chiral symmetry breaking should produce $8 + 8 - 8 = 8$ NGBs, exactly the octet we were expecting.

As we saw in the previous section, the NGBs can be associated with the exponential matrix

$$u(\pi) = e^{\frac{i\pi^a(x)B^A}{\sqrt{2}f}}, \quad (6.25)$$

where B^A are the broken generators. For chiral symmetry breaking it can be shown that the coset $SU(3)_L \times SU(3)_R/SU(3)_V$ is isomorphic to $SU(3)$, so that we can identify the generators B^A with the Gell-Mann matrices.

However, unlike what happens in perturbative spontaneous symmetry breaking,

there is no notion of potential here, and we have no way to compute the value of the vev from the QCD parameters. How should we then write a Lagrangian for the NGBs? It was shown in two seminal papers by Callan-Coleman-Wess-Zumino (CCWZ) already in 1968-1969 that the only object we need to construct the Lagrangian for the NGBs is the matrix $u(\pi)$ of Eq. (6.25), since the rest follows from symmetry arguments.

Let us sketch the CCWZ argument. The main observation is that, at least around the identity, any element $g \in G$ can be written as

$$g = e^{i\pi^\alpha B^\alpha / f} e^{i\alpha^A U^A} = u(\pi) h . \quad (6.26)$$

Let us now focus on u and apply a G transformation. Using the form above we have

$$u(\pi) \rightarrow gu(\pi) = g' = u(\pi')h(g, \pi) , \quad (6.27)$$

where we have denoted explicitly that the decomposition after the transformation depends on the transformed NGBs π' , on the old NGBs π and on the transformation applied g . We thus deduce immediately

$$u(\pi') = gu(\pi)h^\dagger(g, \pi) , \quad (6.28)$$

where g is a global transformation while h is a local transformation, since it depends on $\pi(x)$. This transformation is clearly non-linear, but it becomes linear once we specialize to $g = h$. In fact, the matrix of NGBs transforms in the adjoint of the H subgroup, when $g = h$.

We can construct an object that transforms simply under the unbroken subgroup H as follows: define an object (called Maurer-Cartan one form)

$$\alpha_\mu = -iu^\dagger(\pi)\partial_\mu u(\pi) \equiv d_\mu + E_\mu \quad (6.29)$$

that lives in the Lie algebra of G (because of the derivative). The d_μ and E_μ objects are defined as the projections along the broken and unbroken generators, respectively:

$$d_\mu = d_\mu^\alpha B^\alpha , \quad E_\mu = E_\mu^A U^A . \quad (6.30)$$

Remembering that the generators can always be chosen such that $\text{tr}(B^A U^B) = 0$, we see that

$$\alpha_\mu \rightarrow \underbrace{h\alpha_\mu h^\dagger}_{G=H+G/H} + \underbrace{ih\partial_\mu h^\dagger}_H , \quad (6.31)$$

where we have denoted explicitly the part of the Lie algebra to which each term belongs.

We thus conclude that

$$\begin{aligned} d_\mu &\rightarrow h d_\mu h^\dagger \\ E_\mu &\rightarrow h E_\mu h + ih\partial_\mu h^\dagger . \end{aligned} \quad (6.32)$$

✚ **Exercise 6.3** Show that expanding the NGB exponential up to $\mathcal{O}(\pi^2)$, we have

$$\begin{aligned} d_\mu^\alpha &= \frac{\partial_\mu \pi^\alpha}{f} - \frac{f^{\rho\sigma\alpha}}{2f^2} \partial_\mu \pi^\rho \pi^\sigma + \dots, \\ E_\mu^A &= -\frac{f^{\alpha\beta A}}{2f^2} \partial_\mu \pi^\alpha \pi^\beta + \dots \end{aligned} \quad (6.33)$$

The lowest dimensional H invariant can be constructed only out of d_μ , and the resulting Lagrangian (called *chiral Lagrangian* at order p^2) reads

$$\mathcal{L} = \frac{f^2}{2} \text{tr}(d_\mu d_\mu). \quad (6.34)$$

Notice that the Lagrangian contains an infinite tower of higher dimensional operators, all suppressed by increasing powers of f . We can also include terms with 4, 6, \dots derivative. We thus obtain an effective theory, where effective has two meanings: (i) it is effective because it is non-renormalizable, and (ii) it is effective because it is an effective description of meson physics based on symmetries rather than on the QCD Lagrangian.

Before concluding this chapter, we discuss two last points: (i) how to make contact with the meson matrix in Eq. (4.35), and (ii) how to include the quark masses in the chiral Lagrangian.

Let us start with the first point. First of all let us notice that the approximate $SU(3)_F$ flavor symmetry we discussed early on is naturally identified with the approximate $SU(3)_V$ in the chiral symmetry approach. Since the symmetry breaking pattern we want to analyze is $SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$, we need to generalize the CCWZ approach to a non-simple group.⁶ This is easily done repeating the above argument for all the factors. We thus define

$$u_L \rightarrow g_L u_L h^\dagger, \quad u_R \rightarrow g_R u_R h^\dagger, \quad (6.35)$$

one for each of the factors in $SU(3)_L \times SU(3)_R$. We also define two different Maurer-Cartan forms,

$$\alpha_\mu^L = -i u_L^\dagger \partial_\mu u_L, \quad \alpha_\mu^R = -i u_R^\dagger \partial_\mu u_R. \quad (6.36)$$

This is useful because taking now the difference

$$\alpha_\mu^{LR} = \alpha_\mu^L - \alpha_\mu^R \quad (6.37)$$

we obtain an object that transforms in the adjoint of H , $\alpha_\mu^{LR} \rightarrow h \alpha_\mu^{LR} h^\dagger$. It is now an easy manipulation with unitary matrices to show that

$$\alpha_\mu^{LR} = -i u_L^\dagger (\partial_\mu U) u_R = i u_R^\dagger (\partial_\mu U^\dagger) u_L, \quad U = u_L u_R^\dagger \rightarrow g_L U g_R^\dagger. \quad (6.38)$$

In the last step we have introduced a matrix that transforms exactly as the meson matrix we discussed early in this chapter when we take an $SU(3)_V$ transformation $g_L = g_R$.

⁶Loosely speaking, non-simple groups are those that can be written as products, like $SU(3)_L \times SU(3)_R$.

We thus identify it as the exponential of the meson matrix in Eq. (4.35). The $\mathcal{O}(p^2)$ Lagrangian thus read

$$\mathcal{L} = \frac{f^2}{2} \text{tr}(\alpha_\mu^{LR} \alpha_\mu^{LR}) = \frac{f^2}{2} \text{tr}(\partial_\mu U^\dagger \partial_\mu U). \quad (6.39)$$

Let us now discuss how to explicitly include quark masses, i.e. an explicit chiral symmetry breaking. The idea is to use again the symmetries of the problem to try to deduce the terms that can contain the quark mass matrix M_q . Before discussing this issue, let us stress that, in complete generality,

the dependence of observables on the parameters of the theory is completely determined by symmetry selection rules apart from $\mathcal{O}(1)$ factors.

A detour: the classical pendulum. Let us give an example from classical physics: the simple pendulum. The pendulum is a system characterized by its mass m , its length ℓ and the dynamics is generated by the gravity acceleration g . It is well known that dimensional analysis gives for the frequency

$$\omega = \mathcal{O}(1) \sqrt{g/\ell}. \quad (6.40)$$

What is less known is that dimensional analysis is simply the set of selection rules due to *dilatations* of some quantity. For instance, for space and time dilatations we have

$$\begin{aligned} \mathbf{x} &\rightarrow \lambda_x \mathbf{x}, \\ t &\rightarrow \lambda_t t. \end{aligned} \quad (6.41)$$

This transformation correspond to the freedom of choosing an arbitrary unit for each of the quantities. Let us look at the Lagrangian of the system:

$$L = \frac{m}{2} \dot{\mathbf{x}}^2 - V = \frac{m\ell}{2} \dot{\theta}^2 - mg\ell(1 - \cos \theta). \quad (6.42)$$

If we apply the dilatation transformation to the kinetic term we obtain

$$\frac{m}{2} \dot{\mathbf{x}}^2 \rightarrow \frac{\lambda_x^2}{\lambda_t^2} \frac{m}{2} \dot{\mathbf{x}}^2. \quad (6.43)$$

Comparing with the right hand side, we see that we can obtain the same transformation rule *provided* $\ell \rightarrow \lambda_x \ell$. We are taking a parameter and assign to it a well-defined transformation rule under the symmetry. If we now require complete covariance of the Lagrangian (i.e. we require the potential term to have the same transformation rule under the symmetry as the kinetic term) we see that we must require

$$g \rightarrow \frac{\lambda_x}{\lambda_t^2} g. \quad (6.44)$$

Again, we take a fixed parameter and assign it fictitious transformation rules. What happens at the level of equation of motion? A quick look at the Euler-Lagrange equations shows that the EoM must be covariant under the dilatation symmetry, hence we must

expect the quantities appearing (the frequency, for instance) to be covariant objects. This fixes univocally the combinations of parameters that can appear.

We now go back to the inclusion of quark masses in the chiral Lagrangian. The idea is exactly the same as in the case of the pendulum: we can assign fictitious transformation rules to the parameters of the theory to formally recover invariance under some symmetry. We then write our “observable” (the effective Lagrangian, in this case) in terms of the parameters respecting the symmetry selection rules. We start from the QCD mass term

$$\mathcal{L}_{QCD, mass} = -\bar{q}_L M_q q_R + h.c. \quad (6.45)$$


We know that this term breaks explicitly $SU(3)_L \times SU(3)_R$. If we however imagine that the mass term transforms as

$$M_q \rightarrow g_L M_q g_R^\dagger, \quad (6.46)$$

chiral invariance is recovered. The idea is thus to promote M_q to a *spurion*, i.e. an object that has a well defined transformation under the symmetry it breaks, and to use the spurion to construct the Lagrangian at the meson level. This means that we can write

$$\mathcal{L} = \frac{f^2}{2} \text{tr}(\partial_\mu U^\dagger \partial_\mu U) + f \mu^2 \text{tr}(U M_q^\dagger) + h.c. \quad (6.47)$$

where μ^2 is a new dimensionful parameter that cannot be fixed by symmetry arguments but must be deduced from experiments. Of course, a priori nothing dictates that this procedure, based on symmetries, should work. Remarkably, however, the predictions for the meson masses that can be extracted from the chiral Lagrangian are in remarkable agreement with the experiments.

 **Exercise 6.4** Using the chiral Lagrangian with M_q compute the meson masses in terms of the quark masses and of the parameters f and μ^2 .

Using spurion techniques we can also include other types of explicit chiral symmetry breaking. One important example are the interactions with the photon. The argument proceeds as follows: imagine we gauge $SU(3)_L \times SU(3)_R$ introducing couplings of the form

$$\mathcal{L}_{int} = \bar{q}_L \gamma^\mu L_\mu q_L + \bar{q}_R \gamma^\mu R_\mu q_R, \quad (6.48)$$

in the QCD Lagrangian. In the case of the photon we know that

$$L_\mu = R_\mu = eQ A_\mu, \quad Q = \text{diag} \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right). \quad (6.49)$$


Since this coupling is not of the form $V_\mu^A \lambda^A$, i.e. cannot be written as an object of

the adjoint of $SU(3)$, it constitutes an explicit breaking of the chiral symmetry. We can recover the global $SU(3)_L \times SU(3)_R$ symmetry promoting L_μ and R_μ to spurions that transform in the adjoint of $SU(3)_L$ and $SU(3)_R$, respectively. But we can do even more: if we promote $SU(3)_L \times SU(3)_R$ to a *local* symmetry, the kinetic part of the QCD Lagrangian is no longer invariant under the local chiral transformation, but the non invariant piece can be reabsorbed using a “gauge transformation” of the L_μ and R_μ objects, whose transformation now becomes

$$L_\mu \rightarrow g_L(x)L_\mu g_L^\dagger(x) + i\partial_\mu g_L(x)g_L^\dagger(x), \quad R_\mu \rightarrow g_R(x)R_\mu g_R^\dagger(x) + i\partial_\mu g_R(x)g_R^\dagger(x). \quad (6.50)$$

We are obtaining a neat result: the effect of L_μ and R_μ can be obtained simply by including them in a covariant derivative defined as

$$D_\mu U = \partial_\mu U + iL_\mu U - iUR_\mu. \quad (6.51)$$


 **Exercise 6.5** Show that the covariant derivative must take the form of Eq. (6.51) for $D_\mu U$ to transform as $D_\mu U \rightarrow g_L(x)D_\mu U g_R(x)$ with L_μ and R_μ transforming as normal gauge fields.

The final Lagrangian is thus given by

$$\mathcal{L} = \frac{f^2}{2} \text{tr}(D_\mu U^\dagger D_\mu U) + f\mu^2 \text{tr}(UM_q^\dagger) + h.c. \quad (6.52)$$

and contains masses and gauge interactions of the mesons.

6.4 Additional exercises

 **Exercise 6.6** Consider an effective interaction between a neutral fermion χ and quarks of the form

$$\mathcal{L} = \frac{c}{\Lambda^2} (\bar{\chi}\gamma^\mu\chi) (\bar{q}\gamma^\mu q), \quad (6.53)$$

where q is the triplet of light quarks. The fermion χ has mass m_χ :

- Compute the color and spin averaged cross section $q\bar{q} \rightarrow \chi\bar{\chi}$;
- Write the chiral Lagrangian including such term. Which mesons decays are generated?

- Although we have not discussed them in these lectures, vector mesons play an important role in low energy QCD. Compute the decay width of the $\rho \rightarrow \chi\bar{\chi}$ decay using the matrix elements listed in the Appendix of [arXiv:2011.04735](#).
-

6.5 Additional readings

- S.Weinberg, “The Quantum Theory of Fields” vol. 1;
- M.Peskin and D.Schroeder, “An Introduction To Quantum Field Theory”;
- S.Coleman, “Lectures on Quantum Field Theory”;
- M.Schwartz, “Quantum Field Theory and the Standard Model”;
- R. Penco, “An Introduction to Effective Field Theories” ([link](#));

Part III

Electroweak interactions: The Standard Model of Particle Physics

Chapter 7 The Standard Model

This chapter will be devoted to the bottom-up construction of the Electroweak Theory, i.e. the theory that unifies weak and electromagnetic interactions. We will start with a sketch of the history from the 1930's to the end of the 1960's, showing how to put together many of the concepts already developed in the previous chapters to properly describe the physics of weak interactions. We will then construct explicitly the Standard Model (the theory of strong and electroweak interactions) and compute its main phenomenological consequences.

7.1 Weak interactions

7.1.1 Weak interactions

In the first half of the XX century the experimental observation of decays whose origin could not be traced to electromagnetic or strong interactions suggested the existence of a new type of interaction. These decays are observed to have much longer lifetimes than decays mediated by strong interactions. Since the lifetime of a process is inversely proportional to its coupling, the new type of interactions were called *weak interactions*. For simplicity, in our discussion we will focus on the following two processes:

$$\begin{aligned}\beta - \text{decay} : n &\rightarrow p e \bar{\nu}_e \\ \mu - \text{decay} : \mu &\rightarrow e \nu_\mu \bar{\nu}_e .\end{aligned}\tag{7.1}$$

The first attempt to describe these 4-fermion processes was made by Fermi already in 1933 via the phenomenological Lagrangians

$$\begin{aligned}\mathcal{L}_\beta &= -\frac{G^{(\beta)}}{\sqrt{2}} [\bar{p}\Gamma n] [\bar{e}\Gamma'\nu_e], \\ \mathcal{L}_\mu &= -\frac{G^{(\mu)}}{\sqrt{2}} [\bar{\nu}_\mu\Gamma\mu] [\bar{e}\Gamma'\nu_e],\end{aligned}\tag{7.2}$$

where, as customary, we have denoted the fields with the same symbol as the particles they are describing. Γ and Γ' are Lorentz structures in the list $\{\mathbf{1}, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}$ chosen in such a way to guarantee Lorentz invariance. Fermi wrote its theory choosing $\Gamma = \mathbf{1} = \Gamma'$, but it was soon realized that different Lorentz structures were possible. Different Lorentz structures correspond to different angular behaviors of the decay width, and can be experimentally tested. An example is shown in Fig. 7.1 for various Γ and Γ' structures. Measuring the number of events as a function of the angle allows to

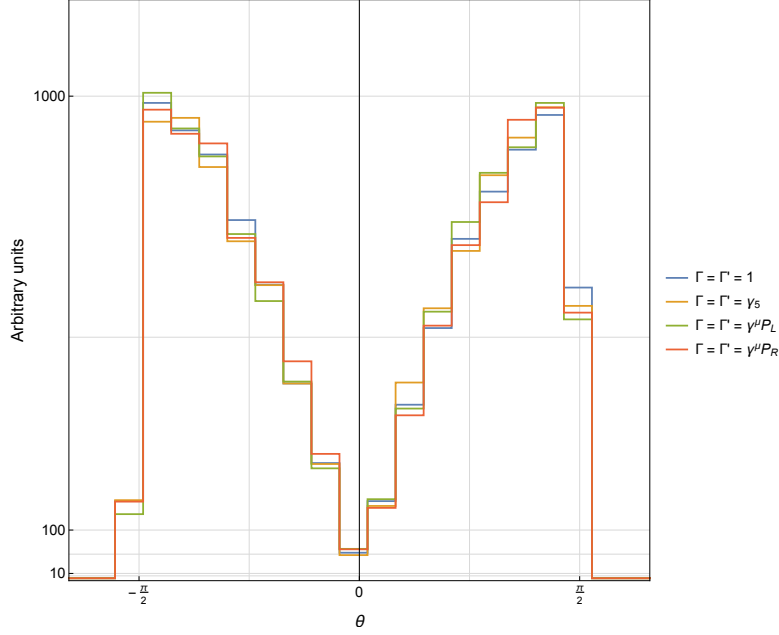


Figure 7.1: Number of events (in arbitrary units) as a function of the electron's angle (measured with respect to some arbitrarily defined direction). As can be seen, different Dirac structures in Eq. (7.2) give different angular dependence that can be experimentally distinguished one from the other. The plot has been generated with MADGRAPH@NLO.

distinguish between various possibilities. Over the two following decades it became clear that the correct form of the Lagrangian is

$$\begin{aligned}\mathcal{L}_\beta &= -\frac{G^{(\beta)}}{\sqrt{2}} [\bar{p}\gamma^\mu(1 - a\gamma_5)n] [\bar{e}\gamma^\mu(1 - \gamma_5)\nu_e], \\ \mathcal{L}_\mu &= -\frac{G^{(\mu)}}{\sqrt{2}} [\bar{\nu}_\mu\gamma^\mu(1 - \gamma_5)\mu] [\bar{e}\gamma^\mu(1 - \gamma_5)\nu_e],\end{aligned}\tag{7.3}$$

where $a = 1.2695 \pm 0.0029$ can be extracted from baryon decays and is a consequence of the composite nature of protons and neutrons. This form of the Lorentz structure is called V-A because it consists of a vector current γ^μ minus an axial current $\gamma^\mu\gamma_5$. This point deserve to be stressed: *only LH fermions interact via weak interactions.*¹

Notice that from dimensional analysis we conclude immediately that $[G^{(\mu)}] = [G^{(\beta)}] = -2$ in energy units. The Fermi Lagrangian is thus non-renormalizable. Moreover, experimentally it was found that $G^{(\mu)} \simeq G^{(\beta)} \equiv G_F \simeq 1.16 \times 10^{-5}$ GeV. The equality between two seemingly unrelated processes suggests a deeper structure in the weak interactions. Interpreting $G_F = 1/\Lambda^2$ as a proxy for the cutoff of the theory, and inserting the experimental value, we obtain

$$\Lambda \sim 300 \text{ GeV} .\tag{7.4}$$

¹Since only LH fermions couple to weak interactions, it would be natural to write the theory in terms of Weyl spinors only, without bothering to work with Dirac fermions. We show in Appendix C how this is done, while we in the main text we use the more common Dirac notation.

As usual, as long as the energy of the process considered is $E \ll \Lambda$ there is no need to worry about whatever physics appears at the cutoff. However, when $E \sim \Lambda$ the predictions of the Fermi theory become completely unreliable, and we need to consider how to UV complete the theory. This is the task we set up to accomplish in this section.

Although a priori there is no reason for the weak interactions to have a behavior similar to QED and QCD, let us suppose the interactions can be written as the coupling between a vector particle and a current. We will follow here loosely the historical developments due to Glashow (1961), Weinberg (1968) and Salam (1969). The discovery of the theory of electroweak interactions won them the Nobel prize in 1979. This is precisely the structure that appears in the theories we already know. Let us start with the leptons. Inspecting Eq. (7.3) we discover that it can be written as the product

$$\mathcal{L}_\mu = -\frac{4G_F}{\sqrt{2}} J_\alpha^{(\mu)} J_\alpha^{(e)\dagger}, \quad J_\alpha^{(f)} = \bar{\nu}_f \gamma_\alpha P_L f = \bar{\nu}_{fL} \gamma_\alpha f_L, \quad (7.5)$$

where $P_L = (1 - \gamma_5)/2$ is the LH projector. Can $J_\alpha^{(i)}$ be written in the form of a current? Clearly it will not be a current like the electromagnetic one, since unlike in QED it involves two different fermions. It can however be written in the form of a non-abelian current $\bar{\psi}_i \gamma^\mu T_{ij}^A \psi_j$, since we know from the discussion in Section 5.2 that such currents connect fermions of different types. Let us thus introduce a doublet

$$L_L \equiv \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad (7.6)$$

where we leave the lepton flavor implicit. We will often simply write $L_L \equiv L$, leaving implicit the fact that this is a doublet of LH fermions. We thus want to write

$$J_\alpha = \bar{L} \gamma_\alpha \tau^+ L, \quad \tau^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (7.7)$$

It is immediate to check that the τ^+ generator does precisely the job we need it to do. Moreover, we recognize τ^+ as an element of the $SU(2)$ algebra, since it can be written in terms of the usual $SU(2)$ generators $\tau^A = \sigma^A/2$ as

$$\tau^+ = \tau^1 + i\tau^2. \quad (7.8)$$

Unexpectedly, it seems that the structure of a non-abelian $SU(2)$ gauge theory is emerging from the known form of the weak interactions. Notice that the conjugate current can be written as

$$J_\alpha^\dagger = \bar{L} \gamma_\alpha \tau^- L, \quad \tau^- = \frac{\tau^1 - i\tau^2}{2}. \quad (7.9)$$

Overall, we are lead to consider two linear combinations of the $SU(2)$ generators to write the weak interactions appearing in muon decay. For compactness of notation, we

will denote such currents as

$$J_\mu^+ = \bar{L}\gamma_\mu\tau^+L, \quad J_\mu^- = \bar{L}\gamma_\mu\tau^-L. \quad (7.10)$$

We are now going to take seriously this hint and suppose that weak interactions are somehow an $SU(2)$ non-abelian gauge theory, with L transforming as a $\mathbf{2}$ of $SU(2)$. Since only LH fermions appear in the current we call this group $SU(2)_L$. Moreover, we *predict* the existence of a third current (not experimentally observed when the theory was written down in the 1960s) whose generator is τ^3 :

$$J_\mu^3 = \bar{L}\gamma_\mu\tau^3L. \quad (7.11)$$

Notice that the J_μ^\pm currents involve one neutral and one charged fermion, in such a way that they have non-trivial transformations under electromagnetism:

$$J_\mu^\pm \rightarrow e^{\mp i\alpha} J_\mu^\pm. \quad (7.12)$$

They are called *charged currents*. On the contrary, the τ^3 generator is diagonal, and it connects fermions of the same charge, with no net charge change. The current J_μ^3 is thus a *neutral current*. The main prediction of the model is thus the existence of neutral currents.

Since the RH fermions e_R (and ν_R in case it exists ³) do not take part in the electroweak interactions they cannot be part of any $SU(2)_L$ current. This leads to the hypothesis that they are $SU(2)_L$ singlets, i.e. they belong to the $\mathbf{1}$ representation.

We now move on and try to write the $SU(2)$ invariant Lagrangian for the lepton doublet. Using our general procedure for non-abelian gauge theories we know that

$$\mathcal{L}_{SU(2)_L} = \bar{L}i\not{D}L, \quad D_\mu L = \partial_\mu L + igW_\mu^A\tau^A L, \quad (7.13)$$

where we called W_μ^A the triplet of $SU(2)_L$ gauge bosons, with $A = 1, 2, 3$. A few points are worth to be stressed:

- Since L transforms as a $\mathbf{2}$ of $SU(2)_L$ and e_R as a $\mathbf{1}$, we cannot combine LH and RH fermions in a mass term. It is thus already clear that additional ingredients will be needed to generate fermion masses;
- Two of the gauge bosons must have non-vanishing electric charge, while the one associated with τ^3 will be neutral. We can write explicitly the charged vectors in terms of W_μ^1 and W_μ^2 comparing with the electric charge of the currents: from Eq. (7.12) we know that J_μ^\pm have electric charge ∓ 1 , respectively; to preserve electromagnetism, we must thus couple J_μ^+ with a vector of electric charge $+1$

²It is worth emphasizing that the \pm superscript refers to the $SU(2)$ generators and not to the electric charge. As a matter of fact, the electric charge is inverted with respect to the \pm .

³As of 2021 there is still no experimental confirmation of the existence of the RH neutrino component.

and J_μ^- with a vector with electric charge -1 . The identification is thus

$$\begin{aligned} W_\mu^1 \tau^1 + W_\mu^2 \tau^2 &= W_\mu^1 \frac{\tau^+ + \tau^-}{2} + W_\mu^2 \frac{\tau^+ - \tau^-}{2i} \\ &= \frac{W_\mu^1 - iW_\mu^2}{2} \tau^+ + \frac{W_\mu^1 + iW_\mu^2}{2} \tau^- . \end{aligned} \quad (7.14)$$

We see immediately that the generators τ^\pm couple to complex combinations of the gauge bosons $W_\mu^{1,2}$ (as it should, since only charged fields can carry electric charge). The normalized states are

$$W_\mu^+ \equiv \frac{W_\mu^1 - iW_\mu^2}{\sqrt{2}}, \quad W_\mu^- \equiv \frac{W_\mu^1 + iW_\mu^2}{\sqrt{2}}, \quad (7.15)$$

in terms of which we can write

$$D_\mu L = \left(\partial_\mu - i \frac{g}{\sqrt{2}} W_\mu^+ - i \frac{g}{\sqrt{2}} W_\mu^- \tau^- + g W_\mu^3 \tau^3 \right) L . \quad (7.16)$$

- We know from QED that (at least) another neutral current must exist in nature, the EM current. It would be tempting to identify our prediction J_μ^3 with J_μ^{EM} . This however cannot work, since we know that the EM current is not chiral (it couples in the same way to LH and RH fermions) and it cannot couple to neutrinos (since they are electrically neutral), while the neutral current of weak interactions have the form

$$J_\mu^3 = \bar{\nu}_L \gamma_\mu \nu_L - \bar{e}_L \gamma_\mu e_L \quad (7.17)$$

and thus couples to neutrinos.

7.1.2 Electroweak unification

We are left with a puzzle: the structure of weak interactions force us to introduce a neutral current J_μ^3 , but this current cannot be identified with J_μ^{EM} . There are two possible ways to proceed: (i) we modify the multiplet structure of the theory extending $SU(2)_L$ to some larger group, trying to identify some of the components with the EM current, or (ii) we admit that we must extend our gauge group to include an additional factor $U(1)$ to account for the EM current. Both ways have been pursued in the literature, and it is ultimately an experimental matter to determine which of the two possibilities is the correct one. In 1973 the first experimental confirmations of a neutral current independent from the EM one was discovered, confirming the second option above. Let us thus follow this path and add a $U(1)$ group to our gauge theory. The first question we need to answer is: should we identify this $U(1)$ with $U(1)_{EM}$? A priori, the answer is no: the gauge theory $SU(2) \times U(1)$ have two neutral currents (one is the non-abelian J_μ^3 and the other one is the one associated with $U(1)$), and the most general possibility is for the EM current to be a linear combination of the two. Let us thus call the new factor

$U(1)_Y$, where Y stands for *hypercharge*.

Our gauge theory will now be

$$G_{EW} = SU(2)_L \times U(1)_Y , \quad (7.18)$$

which is called *electroweak (EW) group*. We need to understand how electromagnetism emerges from this picture. First of all, we observe that since $SU(2)_L$ acts in different ways on the LH and RH components of a fermion, it is quite natural to assume that we can assign different hypercharges to the LH and RH components. This means that our theory will be *chiral*, i.e. LH and RH fermion components will have different charges under G_{EW} .

Since we are now gauging also $U(1)_Y$ a new vector boson must be added to the spectrum, which we will call B_μ . A generic covariant derivative will thus be written as

$$D_\mu = \partial_\mu + igW_\mu^A T^A + ig' B_\mu Y , \quad (7.19)$$

where T^A are the $SU(2)_L$ generators in the appropriate representation (i.e. $T^A = \tau^A$ for the doublets and $T^A = 0$ for the singlets) and Y is a matrix proportional to the identity (because it is associated with a $U(1)$ group that must commute with $SU(2)_L$). We will write it as $Y = y\mathbf{1}$. The Lagrangian for a generic multiplet can be written as

$$\begin{aligned} \mathcal{L} = & i\bar{\psi}\not{\partial}\psi - \frac{g}{\sqrt{2}}W_\mu^+ \bar{\psi}\gamma^\mu T^+ \psi - \frac{g}{\sqrt{2}}W_\mu^- \bar{\psi}\gamma^\mu T^- \psi \\ & - gW_\mu^3 \bar{\psi}\gamma^\mu T^3 \psi - g' B_\mu \bar{\psi}\gamma^\mu Y \psi . \end{aligned} \quad (7.20)$$

We will now assign the quantum numbers Y to guarantee that the EM current appears. To do this, we admit that the physical photon A will be a linear combination of W^3 and B and we call Z the orthogonal combination. They are defined via

$$B = \cos \theta_W A - \sin \theta_W Z , \quad W^3 = \sin \theta_W A + \cos \theta_W Z , \quad (7.21)$$

where θ_W is called weak (or Weinberg) angle. A priori the value of θ_W is completely free, and it must be fixed by experiments. As we are going to see in next chapter, θ_W is small but not negligible, i.e. the physical photon is not a pure B as we could have initially imagined. In terms of A and Z the neutral current Lagrangian can be rewritten as

$$\mathcal{L}_{NC} = \bar{\psi}\gamma^\mu (g \sin \theta_W T^3 + g' \cos \theta_W Y) \psi A_\mu + \bar{\psi}\gamma^\mu (g \cos \theta_W T^3 - g' \sin \theta_W Y) \psi Z_\mu . \quad (7.22)$$

To identify the first term with the EM current we need

$$g \sin \theta_W T^3 + g' \cos \theta_W Y = eQ , \quad (7.23)$$

where Q is the electric charge. Notice that Y always appears in the combination $g'Y$, in such a way that we have the freedom to rescale all the hypercharges by a common factor,

provided we rescale g' by the inverse factor. We can use this freedom to fix arbitrarily one of the hypercharges. Conventionally we choose

$$y_L = -\frac{1}{2}. \quad (7.24)$$

Using this in the definition of the electric charge, Eq. (7.23), we obtain

$$\begin{pmatrix} \frac{g \sin \theta_W - g' \cos \theta_W}{2} & 0 \\ 0 & -\frac{g \sin \theta_W + g' \cos \theta_W}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -e \end{pmatrix}. \quad (7.25)$$

We deduce

$$\tan \theta_W = \frac{g'}{g} \quad \text{and} \quad e = \frac{gg'}{\sqrt{g^2 + g'^2}}. \quad (7.26)$$

We thus get a prediction: if we can determine the gauge couplings g and g' from experiments then we can determine the value of the electric charge. Inserting these results in the general definition Eq. (7.23) we can determine the charge matrix as

$$Q = T^3 + Y. \quad (7.27)$$

Using this formula we can immediately compute the hypercharge of the $SU(2)_L$ singlets: from $T^3 = 0$ we deduce that $Y_{f_R} = Q_{f_R}$, i.e. for RH fermions the hypercharge is indeed the electric charge.

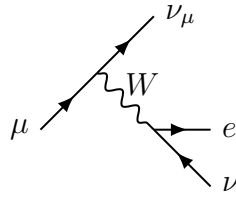
Notice now that nothing of what we said depended on the family of leptons. As a matter of fact, Eq. (7.5) tells us that the $SU(2)_L$ structure is the same for the leptons of the first two generations, and nothing suggests that our construction cannot be repeated *verbatim* to any lepton family (including the third one). Once more, this is ultimately an experimental fact, and we accept that experiments have confirmed until now that the EW interactions obey *lepton universality*, i.e. the representations of the lepton doublets and singlets are independent on the generation,

$$L_i \sim \mathbf{2}_{-1/2}, \quad e_R^i \sim \mathbf{1}_{-1}, \quad i = e, \mu, \tau. \quad (7.28)$$

Before extending our discussion to quarks (and, ultimately, to hadrons), a further essential point must be discussed: how do we obtain the Fermi Lagrangian Eq. (7.5) at low energy starting from the $SU(2)_L \times U(1)_Y$ gauge Lagrangian? It is clear that the Fermi Lagrangian can be obtained by a Feynman diagram in which the W boson is exchanged. Let us write the amplitude for $\mu \rightarrow e \nu_\mu \bar{\nu}_e$ using the gauge Lagrangian:

$$\begin{aligned} \begin{array}{c} \nu_\mu \\ \nearrow \\ \mu \text{ --- } W \\ \searrow \\ e \\ \nearrow \\ \nu_e \end{array} &= \mathcal{M}_{gauge}(\mu(p) \rightarrow e(p_1) + \nu_\mu(p_2) + \bar{\nu}_e(p_3)) \\ &= \frac{g^2}{2} [\bar{u}_{p_2} \gamma^\mu P_L u_p] \frac{-1}{(p - p_2)^2} \eta_{\mu\nu} [\bar{u}_{p_1} \gamma^\nu P_L v_{p_3}] \end{aligned} \quad (7.29)$$

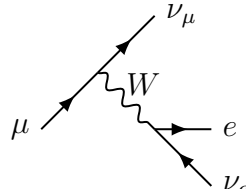
The amplitude we obtain from the Lagrangian in Eq. (7.5) is instead



$$\begin{aligned}
 &= \mathcal{M}_{Fermi}(\mu(p) \rightarrow e(p_1) + \nu_\mu(p_2) + \bar{\nu}_e(p_3)) \\
 &= \frac{4G_F}{\sqrt{2}} [\bar{u}_{p_2} \gamma^\mu P_L u_p] [\bar{u}_{p_1} \gamma^\mu P_L v_{p_3}]
 \end{aligned} \tag{7.30}$$

The fermion structure is the correct one, but the amplitude \mathcal{M}_{gauge} does not produce for us the typical cutoff of the Fermi theory, $\Lambda \sim G_F^{-1/2}$, since the denominator can be made smaller and smaller as the energy becomes smaller and smaller. We conclude that the W boson *cannot be massless*, if we are to obtain the Fermi theory as the low energy limit of our gauge theory. Stated in another way: there is no natural mass scale associated with a massless vector, while the Fermi Lagrangian suggests that the scale $\Lambda \sim G_F^{-1/2}$ should somehow emerge from the gauge theory.

The solution is easy: we need a vector boson to implement the gauge theory, but we need it to be massive. In this way, there is a natural mass scale associated with it, and we have a chance of being able to correctly reproduce the Fermi theory. Ignoring for the moment the problem of how to do so in a consistent way (we will come back to this issue in Sec. 7.2), let us check whether our intuition works. Using the massive vector propagator in Eq. (2.66) the gauge amplitude becomes



$$\begin{aligned}
 &= \mathcal{M}_{gauge}(\mu(p) \rightarrow e(p_1) + \nu_\mu(p_2) + \bar{\nu}_e(p_3)) \\
 &= \frac{g^2}{2} [\bar{u}_{p_2} \gamma^\mu P_L u_p] \frac{-1}{(p-p_2)^2 - m_W^2} \left(\eta_{\mu\nu} - \frac{(p-p_2)_\mu (p-p_2)_\nu}{m_W^2} \right) [\bar{u}_{p_1} \gamma^\nu P_L v_{p_3}] \\
 & \quad [\text{take the low energy limit } (p-p_2)^2 \ll m_W^2] \\
 &\simeq \frac{g^2}{2m_W^2} [\bar{u}_{p_2} \gamma^\mu P_L u_p] [\bar{u}_{p_1} \gamma^\mu P_L v_{p_3}]
 \end{aligned} \tag{7.31}$$

which is exactly of the form of the Fermi amplitude once we identify

$$\frac{4G_F}{\sqrt{2}} = \frac{g^2}{2m_W^2}. \tag{7.32}$$

We conclude that the $SU(2)_L \times U(1)_Y$ gauge theory we have developed in this section can be compatible with the low energy Fermi theory provided we find a way to give mass to the W boson. Surprisingly, such phenomenon is connected with the spontaneous symmetry breaking phenomenon discussed in Sec. 6.2.

7.2 How to give mass to a gauge boson

As we saw in the previous section, we need to add a W mass term to the $SU(2)_L \times U(1)_Y$ gauge Lagrangian. To be more specific, also the Z boson must become massive, to avoid the problem of unseen low energy neutral currents which do not have the right low energy limit. The Lagrangian of a massive vector was already presented in Eq. (1.45), so that naively it looks like it is sufficient to add by hand a W and Z mass term to the gauge Lagrangian to obtain the desired result. Things are not so simple, however. Our whole discussion have been based on the Yang-Mills construction in which the W^A and B vectors shift under a gauge transformation,

$$\begin{aligned} W_\mu^A &\rightarrow W_\mu^A - \frac{1}{g} \partial_\mu \alpha^A - \epsilon^{ABC} \alpha^B W_\mu^C, \\ B_\mu &\rightarrow B_\mu - \frac{1}{g'} \partial_\mu \beta, \end{aligned} \quad (7.33)$$

where we explicitly used the $SU(2)_L$ structure constants $f^{ABC} = \epsilon^{ABC}$ and we called α^A and β the parameters of a $SU(2)_L \times U(1)_Y$ transformation. A mass term added by hand in the Lagrangian is clearly not gauge invariant, casting doubts over our whole procedure. The question we want to address in this section is: can we give mass to the gauge bosons *without spoiling gauge invariance*?

Let us discuss before the case of a $U(1)$ gauge boson. We would like to be able to write the Lagrangian (see App. A)

$$\mathcal{L} = \mathcal{L}_{matter} - \frac{1}{4} (V_{\mu\nu})^2 + \frac{m^2}{2} (V_\mu)^2, \quad (7.34)$$

in a way which is invariant under a gauge transformation. A problem which is not apparent from the previous Lagrangian is that *the theory of a massive vector boson is not well behaved at high energies*. An heuristic argument to see why this happens is to remember that the massive vector boson field obeys $\partial \cdot V = 0$. Let us see what this means at the level of polarization vectors. Aligning the z -axis along the momentum direction, $k^\mu = (E, 0, 0, k)$, we obtain the three independent transverse polarizations

$$\epsilon_1 = (0, 1, 0, 0), \quad \epsilon_2 = (0, 0, 1, 0), \quad \epsilon_3 = \left(\frac{k}{m}, 0, 0, \frac{E}{m} \right). \quad (7.35)$$

The polarization vectors satisfy $k \cdot \epsilon_i = 0$ and $\epsilon_i^2 = -1$. When $E \gg m$ the third polarization vector tends to

$$\epsilon_3 \rightarrow \left(\frac{E}{m}, 0, 0, \frac{E}{m} \right) \quad (7.36)$$

and dominates over the other two because of the large ratio E/m . Processes with external vector bosons will thus grow with E^2/m^2 , which no matter how small the coupling with matter is, diverges at very high energies, destabilizing the unitarity of the

theory. Another aspect of this problem can be seen from the vector boson propagator

$$\frac{-i}{k^2 - m^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2} \right). \quad (7.37)$$

Again we see that for very high energies $k \gg m$ the first term vanishes, while the second one tends to become a constant. This also spoils the high energy behavior of the theory, because when the propagator is embedded in a loop it receives contributions that are UV sensitive. The theory of a massive vector is thus an EFT.

What we want to do is to find a way to recover gauge invariance in the theory and, in doing so, make the massive vector phenomenology well behaved at high energies. The trick (due to Stueckelberg) to achieve gauge invariance is to decompose the vector field as

$$V_\mu = A_\mu - \frac{\partial_\mu \pi}{f}. \quad (7.38)$$

We have inserted the massive scale f for dimensional consistency. This decomposition clearly has some sort of gauge invariance through the transformation

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{\partial_\mu \xi(x)}{g}, \quad \pi(x) \rightarrow \pi(x) + \frac{f}{g} \xi(x). \quad (7.39)$$

We have already encountered scalar fields that shift under a symmetry: in the case of a global symmetry, they were the NGBs associated with the spontaneous breaking of that symmetry (see Sec. 6.2 and, more specifically, Eq. (6.17)). This heuristic argument leads us to suspect that *if the global symmetry associated with gauge invariance is spontaneously broken, then the gauge bosons may become massive “absorbing” the NGBs as their longitudinal degrees of freedom*. This phenomenon is called the -Brout-Englert-Higgs-Anderson phenomenon (Englert and Higgs were awarded the Nobel prize in 2013 for this discovery). Let us see how this construction cures the bad high energy behavior described above. The massive Lagrangian of Eq. (7.34), written in terms of A_μ and π , is given by

$$\mathcal{L} = -\frac{1}{4}(A_{\mu\nu})^2 + \frac{m^2}{2}(A_\mu)^2 + \frac{1}{2} \frac{m^2}{f^2} (\partial_\mu \pi)^2 + m^2 \partial_\mu \pi A_\mu. \quad (7.40)$$

The previous Lagrangian has two interesting features: (i) we need $f = m$ to guarantee a canonical kinetic term for $\pi(x)$, and (ii) we observe the presence of a strange “kinetic mixing” $\partial_\mu \pi A_\mu$ whose interpretation is problematic. The correct solution to the problem of the mysterious kinetic mixing was given by ’t Hooft in 1971: since the theory is gauge invariant and A_μ has the same transformation as a massless vector, let us add a suitably chosen gauge fixing term to \mathcal{L} . The choice

$$\mathcal{L}_{gf} = -\frac{1}{2\xi} (\partial_\mu A^\mu + m\xi\pi)^2 \quad (7.41)$$

has the virtue that it eliminates completely the kinetic mixing from the Lagrangian. With

the addition of such gauge fixing we can compute the A_μ and π propagators:

$$\begin{aligned} (A_\mu) \quad & \frac{-i}{k^2 - m^2} \left(g^{\mu\nu} - (1 - \xi) \frac{k^\mu k^\nu}{k^2 - \xi m^2} \right) \\ (\pi) \quad & \frac{i}{k^2 - \xi m^2}. \end{aligned} \tag{7.42}$$

We see that the A_μ propagator now vanishes in the $k \gg m$ limit, at least as long as $\xi \neq \infty$. We thus recover the good high energy behavior of the theory as we wanted. The price we have to pay is to have a gauge dependent π propagator, signaling that the π mass term is a complete gauge artifact. This is consistent with the fact that, after all, we are still describing a massive vector field with three physical components. The fact that we find it convenient to decompose it in terms of the (A_μ, π) fields is a useful trick that allows us to recover a good high energy behavior and to see clearly the connection with spontaneous symmetry breaking.

To confirm the connection with spontaneous symmetry breaking let us analyze scalar QED when we have spontaneous symmetry breaking. The Lagrangian will be invariant under the gauge transformation

$$\begin{aligned} \phi & \rightarrow e^{i\xi(x)} \phi(x) \\ A_\mu & \rightarrow A_\mu - \frac{1}{g} \partial_\mu \xi(x), \end{aligned} \tag{7.43}$$

but we assume that the vacuum spontaneously breaks the $U(1)$ symmetry. As we did in Sec. 6.2 we parametrize the scalar field as

$$\phi(x) = e^{i\pi(x)/(\sqrt{2}v)} \left(v + \frac{h}{\sqrt{2}} \right). \tag{7.44}$$

Applying a gauge transformation with parameter $\xi(x) = -\pi(x)/(\sqrt{2}v)$ we obtain

$$\begin{aligned} \phi & \rightarrow v + \frac{h}{\sqrt{2}}, \\ A_\mu & \rightarrow A_\mu - \frac{1}{\sqrt{2}gv} \partial_\mu \pi(x) \equiv V_\mu. \end{aligned} \tag{7.45}$$

This is precisely of the form of Eq. (7.38) once we identify $f = \sqrt{2}gv$. This confirms our suspicion: in a gauge theory we can always eliminate the NGBs from the scalar sector making them explicitly the longitudinal degree of freedom of the vector field, which can now acquire mass. Since in this case $\pi(x)$ does not appear as a massless state in the spectrum it is customary to call it “would-be NGB” or “eaten-up NGB”. To compute the

vector mass we focus on the covariant derivative part of the Lagrangian:

$$\begin{aligned}
|D_\mu\phi|^2 &= |\partial_\mu\phi + igV_\mu\phi|^2 \\
&= |\partial_\mu\phi|^2 + g^2(V_\mu)^2|\phi|^2 - igV_\mu(\partial_\mu\phi^\dagger\phi - \phi^\dagger\partial_\mu\phi) \\
&= \frac{1}{2}(\partial_\mu h)^2 + g^2(V_\mu)^2\left(v + \frac{h}{\sqrt{2}}\right)^2 \\
&= \frac{1}{2}(\partial_\mu h)^2 + \frac{2g^2v^2}{2}(V_\mu)^2 + \sqrt{2}g^2v(V_\mu)^2h + \frac{g^2}{2}(V_\mu)^2h^2.
\end{aligned} \tag{7.46}$$

As in scalar QED without symmetry breaking we have trilinear and quadrilinear interactions hVV and $hhVV$, but we also have explicitly a vector mass term. Comparing with the expression for f found above, we conclude $f = m_V$. The gauge in which the would-be NGBs completely disappear from the Lagrangian is called *unitary gauge* because it makes explicit the particle spectrum of the theory. In this gauge, however, it is not clear how the bad high energy behavior described above is cured.

7.3 The Higgs boson

The conclusion of the previous section is clear: we can obtain massive W and Z bosons in the $SU(2)_L \times U(1)_Y$ theory provided we add a scalar which breaks spontaneously the EW symmetry. Since we observe a massless photon in nature, the $U(1)_{EM}$ must survive symmetry breaking and cannot be broken by the vacuum. The symmetry breaking pattern we need to consider is thus

$$SU(2)_L \times U(1)_Y \rightarrow U(1)_{EM}. \tag{7.47}$$

Since this pattern represents the spontaneous breaking of the electroweak symmetry it is often called “electroweak symmetry breaking” (EWSB). Let us count the number of broken generators: since $SU(2)_L$ has 3 generators while the $U(1)$ groups have each one generator, we conclude that we have $(3 + 1) - 1 = 3$ broken generators. This number beautifully matches the number of massive gauge bosons we want to obtain (two charged and one neutral), suggesting that there is no need to enlarge the gauge group to obtain a UV completion of the Fermi theory.

What about the scalar? There is a certain degree of arbitrariness in the choice of the scalar properties. For sure it must contain at least three real scalars to provide for the three would-be NGBs, and it must have non-trivial transformations under both $SU(2)_L$ and $U(1)_Y$ to ensure the symmetry breaking pattern. The smallest non-trivial $SU(2)_L$ representation is the doublet $\mathbf{2}$ while to allow for a non-trivial hypercharge we must take each of the doublet components complex. The minimal representation for the scalar

boson will thus be

$$H \sim \mathbf{2}_{y_H} . \quad (7.48)$$

where the hypercharge is still unfixed. The scalar H is called the *Higgs doublet*. Notice that H contains 4 real scalar degrees of freedom, hence after symmetry breaking we expect one of real scalar to be still part of the spectrum, since it will not be absorbed by the vector bosons. This scalar, which we will call h , is the *Higgs boson* discovered in 2012 at the CERN-LHC. Let us denote for the moment

$$H = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} . \quad (7.49)$$

We know that H must develop a non-trivial vev to break spontaneously the EW symmetry to QED. In general the vacuum will populate both doublet entries. We remind the reader, however, that all the directions of the vacuum are equivalent, and that we can use an EW transformation to align it along any direction we find convenient. We apply the following transformations:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \xrightarrow{SU(2)_L} \begin{pmatrix} 0 \\ ve^{i\alpha} \end{pmatrix} \xrightarrow{U(1)_Y} \begin{pmatrix} 0 \\ v \end{pmatrix} \equiv \langle H \rangle , \quad v = \sqrt{v_1^2 + v_2^2} . \quad (7.50)$$

By assumption, a $U(1)_{EM}$ transformation must leave the vacuum invariant

$$e^{i\alpha Q_H} \langle H \rangle = \langle H \rangle \quad \Rightarrow \quad Q_H \langle H \rangle = 0 \quad \rightarrow \quad \begin{pmatrix} \frac{1}{2} + y_H & 0 \\ 0 & -\frac{1}{2} + y_H \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \quad (7.51)$$

This selects $y_H = 1/2$ as the Higgs doublet hypercharge. We conclude that the minimal particle content that ensures EWSB is

$$H \sim \mathbf{2}_{1/2} . \quad (7.52)$$

The exponential parametrization of the Higgs doublet must be given in terms of the broken generators, of which we now deduce the explicit form. To do so, we apply a gauge transformation to $\langle H \rangle$ trying to single out the broken generators. We obtain

$$\begin{aligned} e^{i(\alpha^A T^A + \beta Y)} \langle H \rangle &= e^{i[\alpha^A T^A + \beta(Q - T^3)]} \langle H \rangle \\ &= (\text{define } \{\xi^1, \xi^2, \xi^3\} = \{\alpha^1, \alpha^2, \alpha^3 - \beta\}) \\ &= e^{i\xi^A T^A + i\beta Q} \langle H \rangle \\ &= e^{i\xi^A T^A} \langle H \rangle \neq 0 \end{aligned} \quad (7.53)$$

This means that we for the purpose of writing the exponential parametrization of the Higgs doublet we can always choose the broken generators as the $SU(2)_L$ generators,

obtaining

$$H = e^{i\pi^A(x)T^A} \begin{pmatrix} 0 \\ v + \frac{h}{\sqrt{2}} \end{pmatrix}. \quad (7.54)$$

As anticipated, the scalar h is the physical Higgs boson.

Before concluding this section, let us comment on an interesting consequence of Eq. (7.52). We have introduced the Higgs doublet as a mean to give mass to the gauge bosons of the theory in a way that does not spoil the gauge invariance procedure. Now that we have the representations of the leptons and of the Higgs doublet, however, we can write down the most general Lagrangian $SU(2)_L \times U(1)_Y$ invariant:

$$\mathcal{L} = \bar{L}i\not{D}L + \bar{e}_Ri\not{D}e_R + |D_\mu H|^2 - V(H) - \bar{L}Y_e e_R H + h.c. \quad (7.55)$$

Notice that in addition to the Higgs doublet potential $V(H)$ (that must guarantee EWSB) there is a term of the form fermion-fermion-scalar called *Yukawa* term. Since it is the first time that we encounter a term like this, let us analyze in detail its invariance. Everything rests on the observation that each of the fields L , e_R and H must be thought as carrying different indices, one for each non-trivial transformation they undergo. For instance, L is a LH Dirac spinor (hence it must carry a Lorentz index a) and is also an $SU(2)_L$ doublet (hence it must carry an index A which represents the two non-trivial entries of the doublet). Analogously, e_R must carry a Lorentz index (but not an $SU(2)_L$ index, since it is a singlet) and H must carry an $SU(2)_L$ index (but not a Lorentz index, since it is a Lorentz scalar). Writing all indices explicitly, we have

$$L_{aA}, \quad e_{R,b}, \quad H_B, \quad (7.56)$$

where capital letters denote the $SU(2)_L$ doublet indices and small letters denote spinor Lorentz indices. Under a Lorentz transformation we have

$$(\text{Lorentz}) \quad L_{aA} \rightarrow S_a^b L_{bA}, \quad e_{R,b} \rightarrow S_b^c e_{R,c}, \quad H_A \rightarrow H_A, \quad (7.57)$$

while under an $SU(2)_L$ transformation we have

$$(SU(2)_L) \quad L_{aA} \rightarrow U_A^B L_{aB}, \quad e_{R,b} \rightarrow e_{R,b}, \quad H_A \rightarrow U_A^B H_B. \quad (7.58)$$

Finally, under an hypercharge transformation we have

$$(U(1)_Y) \quad L_{aA} \rightarrow e^{-i\alpha/2} L_{aA}, \quad e_{R,b} \rightarrow e^{-i\alpha} e_{R,b}, \quad H_A \rightarrow e^{i\alpha/2} H_A. \quad (7.59)$$

We are now in a position to check the overall invariance of the Yukawa term. As we did discussing the complex conjugate representation in Sec. 4.2, complex conjugation corresponds to raising the index. Moreover, we remind the reader that the combination $\bar{L} = L^\dagger \gamma^0$ transform as

$$\bar{L}^{aA} \rightarrow \bar{L}^{bA} (S^{-1})_b^a. \quad (7.60)$$

We obtain

$$\begin{aligned}
\bar{L}^{aA} e_{R,a} H_A &\rightarrow \bar{L}^{bB} (S^{-1})_b^a U_B^A e^{i\alpha/2} S_a^c e^{-i\alpha} e_{R,c} U_A^C e^{i\alpha/2} H_C \\
&= \underbrace{(S^{-1})_b^a S_a^c}_{\delta_b^c} \underbrace{U_B^A U_A^C}_{\delta_B^C} \underbrace{e^{i\alpha/2} e^{-i\alpha} e^{i\alpha/2}}_1 \bar{L}^{bB} e_{R,c} H_C \\
&= \bar{L}^{aA} e_{R,a} H_A
\end{aligned} \tag{7.61}$$

which is invariant as expected. From the practical point of view we do not need to go through this process every time. To check the invariance of any term in the Lagrangian we can proceed as follows: (i) we only write Lorentz invariant terms to begin with, (ii) we check the invariance of the term with respect to the gauge transformation. Let us apply this procedure to the Yukawa term. Given our particle content L , e_R and H there are two Lorentz invariant terms that we can write:

$$\bar{L} e_R H \quad \text{and} \quad \bar{L} e_R H^* . \tag{7.62}$$

Under $SU(2)_L$ we have $\bar{L} \rightarrow \bar{L} U^\dagger$ and $H \rightarrow UH$. What are the invariants that can be constructed out of two $SU(2)_L$ fundamentals? For sure $\bar{L}H \rightarrow \bar{L}U^\dagger UH = \bar{L}H$ is invariant. However, one more invariant can be constructed, remembering that the ϵ_{AB} tensor is a $SU(2)_L$ invariant:

$$\epsilon^{AB} L_A H_B \rightarrow \underbrace{\epsilon^{AB} U_A^C U_B^D}_{\epsilon^{CD}} L_C H_D . \tag{7.63}$$

This invariant can be written in matrix form as $L^T \epsilon H$. Taking the complex conjugate we conclude that $L^\dagger \epsilon H^*$ is also a $SU(2)_L$ invariant. This means that we can define an ‘‘epsilon-contracted’’ doublet $\tilde{H} \equiv \epsilon H^*$ and write the two terms as

$$\bar{L} e_R H, \quad \bar{L} e_R \tilde{H} . \tag{7.64}$$

We stress once more that both terms are Lorentz and $SU(2)_L$ invariant by construction. We now only need to check hypercharge invariance: for $\bar{L} e_R H$ we have $+1/2(\bar{L}) - 1(e_R) + 1/2(H) = 0$, while for $\bar{L} e_R \tilde{H}$ we have $+1/2(\bar{L}) - 1(e_R) - 1/2(\tilde{H}) = -1 \neq 0$, selecting $\bar{L} e_R H$ as the only Lorentz and gauge invariant term that can be constructed in the Lagrangian.

What is the meaning of the Yukawa term? Let us focus on the case of one generation, in which Y_e is simply a complex number (we will come back to the three generation case

in Sec. 7.5). In the unitary gauge we have

$$\begin{aligned}
\mathcal{L}_{Yukawa} &= Y_e \bar{L} e_R H \\
&= Y_e \begin{pmatrix} \bar{\nu}_L \\ \bar{e}_L \end{pmatrix} e_R \begin{pmatrix} 0 \\ v + \frac{h}{\sqrt{2}} \end{pmatrix} \\
&= Y_e \bar{e}_L e_R \left(v + \frac{h}{\sqrt{2}} \right).
\end{aligned} \tag{7.65}$$

We see that the term proportional to v provides a *mass term for the electron*, while the term proportional to h is a trilinear interaction of the form h -fermion-fermion. We thus conclude that *the Higgs boson is responsible not only for the gauge boson masses, but for the lepton masses as well.*

7.4 Including quarks

The inclusion of quarks was historically more challenging, since as we saw there are no free quarks in nature. Instead of going through the history of how quarks were introduced in the game, let us interpret the β -decay process $n \rightarrow pe\bar{\nu}_e$ in terms of quarks and from there deduce the properties we need. Remember that the quark content of a neutron is udd while the one of a proton is uud . It is thus immediate to conclude that in β -decay what happens is that a d -quark in the neutron becomes a u -quark in the proton, with the other two quarks being simple spectators of the process. This means that we can write the β -decay Fermi Lagrangian as

$$\mathcal{L}_\beta = -\frac{G^{(\beta)}}{\sqrt{2}} [\bar{u}\gamma^\mu(1 - \gamma_5)d] [\bar{e}\gamma^\mu(1 - \gamma_5)\nu_e]. \tag{7.66}$$

The only non obvious thing is what happens with the non-perturbative a factor of Eq. (7.3). Since it is a factor that takes into account the non-perturbative QCD interactions that bind the proton and the neutron we will simply assume that the underlying interactions at the quark level satisfy the $V - A$ structure already discussed for leptons. Once more, only experiments can determine whether this assumption is correct. No experimental deviation from the consequences of this hypothesis has been observed so far.


Eq. (7.66) has exactly the same structure as the lepton part of Eq. (7.3). Motivated by this correspondence, we assemble the LH and RH quarks in analogy with what we have done for leptons:

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \equiv Q \sim \mathbf{2}_{y_Q}, \quad u_R \sim \mathbf{1}_{y_u}, \quad d_R \sim \mathbf{1}_{y_d}. \tag{7.67}$$

What is left to do is determine the hypercharges of the fields. Proceeding as with the

leptons we find immediately that

$$y_Q = \frac{1}{6}, \quad y_u = \frac{2}{3}, \quad y_d = -\frac{1}{3}. \quad (7.68)$$

 **Exercise 7.1** Show explicitly that the values of the hypercharges listed above are the correct ones.

7.5 The Standard Model of Particle Physics

We are finally in a position to write down the complete SM Lagrangian. First of all we collect the representations of the fields. These representations are independent on the family, i.e. all the families (both in the lepton and quark sector) have exactly the same gauge representations. In the case of the quark we remember that they transform non-trivially also under the QCD group $SU(3)_c$, so that we define the SM as a gauge theory based on

$$SU(3)_c \times SU(2)_L \times U(1)_Y. \quad (7.69)$$

We write the fields representations as $(\mathbf{R}_{SU(3)_c}, \mathbf{R}_{SU(2)_L})_y$:

$$\begin{aligned} \text{leptons} \quad & L^i \sim (\mathbf{1}, \mathbf{2})_{-1/2}, \quad e_R^i \sim (\mathbf{1}, \mathbf{1})_{-1} \\ \text{quarks} \quad & Q^i \sim (\mathbf{3}, \mathbf{2})_{1/6}, \quad u_R^i \sim (\mathbf{3}, \mathbf{1})_{2/3}, \quad d_R^i \sim (\mathbf{3}, \mathbf{1})_{-1/3} \end{aligned} \quad (7.70)$$

We have explicitly added a family index $i = 1, 2, 3$ to stress that the structure repeats across families. The Standard Model (SM) Lagrangian then reads

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{fermion} + \mathcal{L}_H + \mathcal{L}_{Yuk}, \quad (7.71)$$

where

$$\begin{aligned} \mathcal{L}_{gauge} &= -\frac{1}{4}(G_{\mu\nu}^A)^2 - \frac{1}{4}(W_{\mu\nu}^a)^2 - \frac{1}{4}(B_{\mu\nu})^2 + \mathcal{L}_{GF+GH}, \\ \mathcal{L}_{fermion} &= \bar{L}^i i \not{D} L^i + \bar{e}_R^i i \not{D} e_R^i + \bar{Q}^i i \not{D} Q^i + \bar{u}_R^i i \not{D} u_R^i + \bar{d}_R^i i \not{D} d_R^i, \\ \mathcal{L}_H &= |D_\mu H|^2 - V(H), \\ \mathcal{L}_{Yuk} &= \bar{L}^i Y_e^{ij} e_R^j H + \bar{Q}^i Y_d^{ij} d_R^j H + \bar{Q}^i Y_u^{ij} u_R^j \tilde{H}. \end{aligned} \quad (7.72)$$

where we have explicitly written the family indices. We now discuss each term in turn.

7.5.1 Gauge Lagrangian

The gauge Lagrangian is given by the gauge bosons kinetic terms, constructed in terms of the field strengths

$$\begin{aligned} G_{\mu\nu}^A &= \partial_\mu G_\nu^A - \partial_\nu G_\mu^A - f^{ABC} G_\mu^B G_\nu^C, \\ W_{\mu\nu}^a &= \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - \epsilon^{abc} W_\mu^b W_\nu^c, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu. \end{aligned} \quad (7.73)$$

The additional term \mathcal{L}_{GF+GH} contains the gauge fixing and ghost terms, needed to ensure the unitarity of the theory.

7.5.2 Fermion Lagrangian

The fermion Lagrangian simply consists of the sum over all possible fermions with the appropriate covariant derivative. For instance, for the $SU(2)_L$ doublets we have

$$\begin{aligned} D_\mu L &= (\partial_\mu + igW_\mu^a T_L^a + ig' B_\mu Y_L) L, \\ D_\mu Q &= (\partial_\mu + ig_s G_\mu^A T_c^A + igW_\mu^a T_L^a + ig' B_\mu Y_Q) Q, \end{aligned} \quad (7.74)$$

where we denote with T_c^A and T_L^a the $SU(3)_c$ and $SU(2)_L$ generators, respectively, and as we already said the matrices Y_L and Y_Q are proportional to the identity. What about the flavor? In writing the SM Lagrangian we have neglected the possibility that the kinetic terms can connect to different flavors. Take the case of the lepton doublet L , for which we can write

$$\bar{L}^i K_L^{ij} i \not{D} L^j \quad (7.75)$$

for some hermitian kinetic matrix K . After all, we have repeatedly stressed that the gauge transformation does not depend on the family. Moreover, we know that the objects in the Lagrangian are not physical, hence the possibility of a non diagonal kinetic term in flavor space cannot be discarded a priori. Suppose we write all our kinetic terms in off-diagonal form in flavor space. Since the matrices K_f are hermitian, we know that they can be diagonalized by a unitary transformation

$$K_f = U_f K_f^{diag} U_f^\dagger, \quad (7.76)$$

where U_f is a unitary matrix in flavor space. Redefining the fields to absorb the unitary matrices we obtain diagonal kinetic terms of the form

$$\bar{f}^i k_f^i i \not{D} f^i, \quad (7.77)$$

where k^i is the K_f eigenvalue. To bring the kinetic term in canonical form we can now redefine $f^i \rightarrow f^i / \sqrt{k^i}$. The main conclusion is that we can always bring the kinetic Lagrangian in a form which is canonical and diagonal in flavor space. We will thus

directly work in this flavor basis from the very beginning. We are going to see later on that more flavor transformations will be needed to go into a basis in which the fields are mass eigenstates.

Before moving on, let us explicitly work out the form of the interactions between the fermions and the gauge bosons. We stress once more that these fermions still do not have a well-defined mass, so these interactions should be considered as an intermediate step towards the computation of the couplings between the physical gauge bosons and the physical fermions. Remembering the definition of charge we write

$$\begin{aligned}
 gW_\mu^A T_L^A + g'B_\mu Y &= \frac{g}{\sqrt{2}}W_\mu^+ T^+ + \frac{g}{\sqrt{2}}W_\mu^- T^- + g(s_W A_\mu + c_W Z_\mu) T_L^3 \\
 &\quad + g'(c_W A_\mu - s_W Z_\mu) Y \\
 &= \frac{g}{\sqrt{2}}W_\mu^+ T^+ + \frac{g}{\sqrt{2}}W_\mu^- T^- + \sqrt{g^2 + g'^2} [s_W c_W (T_L^3 + Y) A_\mu \\
 &\quad + (c_W^2 T_L^3 - s_W^2 Y) Z_\mu] \\
 &= \frac{g}{\sqrt{2}}W_\mu^+ T^+ + \frac{g}{\sqrt{2}}W_\mu^- T^- + eQ A_\mu + \sqrt{g^2 + g'^2} (T_L^3 - s_W^2 Q) Z_\mu.
 \end{aligned} \tag{7.78}$$

We have denoted with s_W and c_W the sine and cosine of the weak angle, and we have used the definitions of electric charge $Q = T_L^3 + Y$ and of the physical A and Z bosons. As expected, the photon couples via eQ . As for the Z boson, notice that s_W^2 measures how much the Z coupling is deformed from a pure $V - A$ current (that would be simply proportional to T_L^3).

7.5.3 Higgs Lagrangian

The Higgs Lagrangian is given by the Higgs doublet kinetic term and by its potential. For our construction to be relevant for the phenomenology of particle physics, we need the potential to have a non-trivial family of vacua to trigger EWSB. It is easier to do the computation directly in matrix form. The kinetic part of the Higgs Lagrangian can be

written in the unitary gauge as

$$\begin{aligned}
 D_\mu H &= \partial_\mu H + igW_\mu^a T_L^A H + ig' B_\mu Y_H H \\
 &= \begin{pmatrix} 0 \\ \frac{\partial_\mu h}{\sqrt{2}} \end{pmatrix} + \frac{i}{2} \begin{pmatrix} gW_\mu^3 + g'B_\mu & \sqrt{2}gW_\mu^+ \\ \sqrt{2}gW_\mu^- & -gW_\mu^3 + g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v + \frac{h}{\sqrt{2}} \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ \frac{\partial_\mu h}{\sqrt{2}} \end{pmatrix} + i \begin{pmatrix} eA_\mu + \sqrt{g^2 + g'^2} \left(\frac{1}{2} - s_W^2\right) Z_\mu & \frac{g}{\sqrt{2}} W_\mu^+ \\ \frac{g}{\sqrt{2}} W_\mu^- & -\frac{1}{2} \sqrt{g^2 + g'^2} Z_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v + \frac{h}{\sqrt{2}} \end{pmatrix} \\
 &= \begin{pmatrix} i \frac{g}{\sqrt{2}} W_\mu^+ \left(v + \frac{h}{\sqrt{2}}\right) \\ \frac{\partial_\mu h}{\sqrt{2}} - \frac{i}{2} \sqrt{g^2 + g'^2} Z_\mu \left(v + \frac{h}{\sqrt{2}}\right) \end{pmatrix}.
 \end{aligned} \tag{7.79}$$

We can see immediately that there is no trace of the photon field in the Higgs covariant derivative. To compute the masses of the gauge bosons and the interactions between the Higgs boson and the gauge vectors we take the squared modulus of the expression above. We obtain

$$\begin{aligned}
 |D_\mu H|^2 &= \frac{(\partial_\mu h)^2}{2} + \left[\frac{g^2}{2} W_\mu^+ W_\mu^- + \frac{g^2 + g'^2}{4} (Z_\mu)^2 \right] \left(v + \frac{h}{\sqrt{2}}\right)^2 \\
 &= \frac{(\partial_\mu h)^2}{2} + \left[\frac{g^2 v^2}{2} W_\mu^+ W_\mu^- + \frac{(g^2 + g'^2) v^2}{4} (Z_\mu)^2 \right] \left(1 + \frac{h}{\sqrt{2}v}\right)^2
 \end{aligned} \tag{7.80}$$

from which we can identify immediately the W and Z masses:

$$m_W^2 = \frac{g^2 v^2}{2}, \quad m_Z^2 = \frac{(g^2 + g'^2) v^2}{2}. \tag{7.81}$$

Eq. (7.80) contains the first *predictions* of the SM: the trilinear and quadrilinear couplings of the form hVV and $hhVV$ are proportional to the vector mass; moreover, we can define the so-called ρ parameter in terms of the W and Z masses as

$$\rho \equiv \frac{m_W^2}{m_Z^2 c_W^2} = 1 \quad (\text{in the SM at tree - level}). \tag{7.82}$$

Exercise 7.2 Suppose that instead of taking $H \sim \mathbf{2}_{1/2}$ we allow for larger representations, $H \sim \mathbf{R}_{y_H}$. What is the value of the ρ parameter in this case? Show that only a handful of representations ensure $\rho = 1$ at tree-level.

We now turn to the analysis of the Higgs doublet potential. It is easy to convince ourselves that the most general gauge invariant potential is

$$V(H) = -\mu_H^2 H^\dagger H + \lambda (H^\dagger H)^2. \tag{7.83}$$

EWSB will happen for $\mu_H^2 > 0$ (we have appropriately chosen the sign in front of the

Higgs quadratic term to be positive to avoid the appearance of minus signs later on in the computation). We can trade the two parameters μ^2 and λ appearing in the Higgs potential for two physical parameters. The convenient choice is to take the vev v and the mass of the Higgs boson $m_h^2 = \partial^2 V / \partial h^2|_{min}$. We obtain

$$v^2 = \frac{\mu_H^2}{2\lambda}, \quad m_h^2 = -\mu_H^2 + 6\lambda v^2, \quad (7.84)$$

a result that can also be stated as

$$\mu_H^2 = \frac{m_h^2}{2}, \quad \lambda = \frac{m_h^2}{4v^2}. \quad (7.85)$$

 **Exercise 7.3** Verify this result.

The potential can be written in a way that makes clearer the non-trivial vacuum structure of the theory. To this end we use the minimum equation to trade $\mu_H^2 = 2\lambda v^2$, obtaining

$$V(H) = -2\lambda v^2 H^\dagger H + \lambda (H^\dagger H)^2 = \lambda (H^\dagger H - v^2)^2 + \text{const} \quad (7.86)$$

where we have added a constant term to complete the square. As long as we do not couple the SM to gravity we are free to shift the value of the zero energy point of the theory as we please. The last form of the potential is particularly convenient to deduce the form of the Higgs boson self interactions:

$$\begin{aligned} V(H) &= \lambda (H^\dagger H - v^2)^2 \\ &= \lambda \left(\frac{h^2}{2} + \sqrt{2}vh \right)^2 \\ &= \frac{\lambda}{4} h^4 + 2\lambda v h^3 + 2\lambda v^2 h^2 \\ &= \frac{m_h^2}{16v^2} h^4 + \frac{m_h^2}{2v} h^3 + \frac{1}{2} m_h^2 h^2. \end{aligned} \quad (7.87)$$

As happened in the case of the gauge bosons, also the Higgs boson interactions are all proportional to its mass. This is another prediction of the SM.

7.5.4 Yukawa Lagrangian

We finally discuss the Yukawa Lagrangian in Eq. (7.72). Notice that this term is already written in the basis in flavor space in which the kinetic terms are canonical and flavor diagonal. In general, the Yukawa matrices $Y_{e,u,d}$ are generic complex matrices, i.e. in general they will not be diagonal. We can however apply unitary flavor transformations to any of these terms to try to make the Yukawa matrices diagonal. Once this is done, the mass terms given by these terms will be diagonal and we will be in the so-called

“fermion mass basis” in which each fermion field has a well defined mass and can excite a physical particle out of the vacuum.

To this end we recall a result from linear algebra: a generic complex matrix can always be decomposed as

$$Y = U_L Y_{diag} U_R^\dagger, \quad (7.88)$$

where Y_{diag} is a diagonal matrix with positive definite entries and $U_{L,R}$ are unitary matrices. In the SM case this decomposition can be applied to the Yukawa matrices in flavor space. What we want to do is to apply a unitary flavor transformation to the SM fermions to try to leave only the diagonal matrix in the Yukawa terms. More specifically we write

$$Y_e = U_{eL} Y_e^{diag} U_{eR}^\dagger, \quad Y_u = U_{uL} Y_u^{diag} U_{uR}^\dagger, \quad Y_d = U_{dL} Y_d^{diag} U_{dR}^\dagger. \quad (7.89)$$

The SM Yukawa term becomes (we use the bold notation to denote vectors in flavor space) ⁴

$$\begin{aligned} \mathcal{L}_{Yuk} &= \bar{L} Y_e e_R H + \bar{Q} Y_d \mathbf{d}_R H + \bar{Q} Y_u \mathbf{u}_R \tilde{H} \\ &= \bar{L} U_{eL} Y_e^{diag} U_{eR}^\dagger e_R H + \bar{Q} U_{dL} Y_d^{diag} U_{dR}^\dagger \mathbf{d}_R H + \bar{Q} U_{uL} Y_u^{diag} U_{uR}^\dagger \mathbf{u}_R \tilde{H}. \end{aligned} \quad (7.91)$$

If we now apply the unitary flavor transformation

$$\mathbf{f}_R \rightarrow U_{fR} \mathbf{f}_R \quad (7.92)$$

on the RH fermions we completely eliminate the unitary matrices on the right of the diagonal Yukawa matrices in all three terms. The same cannot be done for the unitary matrices on the left of the diagonal Yukawa matrices, however, since there are only two doublets available (L and Q). The lepton term is not a problem, since we can perform the transformation

$$\mathbf{L} \rightarrow U_{eL} \mathbf{L} \quad (7.93)$$

that completely diagonalizes the lepton Yukawa term. For the quark terms we choose to transform

$$\mathbf{Q} \rightarrow U_{uL} \mathbf{Q} \quad (7.94)$$

making the last term diagonal. Our final result is

$$\mathcal{L}_{Yuk} = \bar{L} Y_e^{diag} e_R H + \bar{Q} U_{uL}^\dagger U_{dL} Y_d^{diag} \mathbf{d}_R H + \bar{Q} Y_u^{diag} \mathbf{u}_R \tilde{H}. \quad (7.95)$$

Notice that since the kinetic term was already proportional to the identity in flavor space

⁴To make the notation clearer, we stress that when flavor indices are not explicitly shown we are considering the vectors in flavor space

$$\mathbf{L} = (L_e, L_\mu, L_\tau), \quad \mathbf{e}_R = (e_R, \mu_R, \tau_R) \quad (7.90)$$

and analogous for the quarks.

it is not modified by any of the unitary transformations performed so far.

Let us now go into the unitary gauge, in which we already know that the particle spectrum is manifest. We have

$$\begin{aligned}\mathcal{L}_{Yuk} &= \bar{\mathbf{L}}Y_e^{diag}e_R H + \bar{\mathbf{Q}}U_{uL}^\dagger U_{dL}Y_d^{diag}\mathbf{d}_R H + \bar{\mathbf{Q}}Y_u^{diag}\mathbf{u}_R \tilde{H} \\ &= \bar{\mathbf{e}}_L Y_e^{diag} e_R \left(v + \frac{h}{\sqrt{2}} \right) + \bar{\mathbf{u}}_L Y_u^{diag} \mathbf{u}_R \left(v + \frac{h}{\sqrt{2}} \right) + \bar{\mathbf{d}}_L (U_{uL}^\dagger U_{dL}) \mathbf{d}_R \left(v + \frac{h}{\sqrt{2}} \right).\end{aligned}\quad (7.96)$$

With the exception of the last term we have obtained that the term proportional to the Higgs vev is diagonal and represent the mass matrix of the leptons and up-type quarks. To bring also the last term in diagonal form we perform one last rotation on the LH down quarks. We first define the so-called Cabibbo-Kobayashi-Maskawa (CKM) matrix

$$V_{CKM} \equiv U_{uL}^\dagger U_{dL}. \quad (7.97)$$

Notice that being the product of unitary matrices, the CKM matrix is itself a unitary matrix. We then apply the unitary transformation

$$\mathbf{d}_L \rightarrow V_{CKM} \mathbf{d}_L \quad (7.98)$$

to bring also the last term in diagonal form. We finally obtain

$$\mathcal{L}_{Yuk} = [\bar{\mathbf{e}}_L M_e e_R + \bar{\mathbf{u}}_L M_u \mathbf{u}_R + \bar{\mathbf{d}}_L M_d \mathbf{d}_R] \left(1 + \frac{h}{\sqrt{2}v} \right) + h.c. \quad (7.99)$$

where we have defined the diagonal fermion mass matrices as

$$M_f = Y_f^{diag} v, \quad f = e, u, d. \quad (7.100)$$

Once more we find that the interaction between the Higgs boson and other particles is proportional to the particle mass. Notice that in the SM neutrinos are massless, since we do not introduce RH neutrinos.

So far we have been only concerned with the transformations needed to bring the Yukawa terms in a diagonal form to identify the fermion masses. We now go back to the other terms in the SM Lagrangian and check if the unitary transformations in flavor space we have applied had some effect. Let us first start summarizing the transformations that lead us to the fermion mass basis:

$$\begin{aligned}e_R &\rightarrow U_{eR} e_R & \mathbf{d}_R &\rightarrow U_{dR} \mathbf{d}_R & \mathbf{u}_R &\rightarrow U_{uR} \mathbf{u}_R \\ \mathbf{L} = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} &\rightarrow U_{eL} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} & \mathbf{Q} = \begin{pmatrix} \mathbf{u}_L \\ \mathbf{d}_L \end{pmatrix} &\rightarrow U_{uL} \begin{pmatrix} \mathbf{u}_L \\ \mathbf{d}_L \end{pmatrix} &\rightarrow \begin{pmatrix} \mathbf{u}_L \\ V_{CKM} \mathbf{d}_L \end{pmatrix}\end{aligned}\quad (7.101)$$

There is clearly no effect on the Higgs Lagrangian, since it does not involve fermions. Also, we have repeatedly insisted on the fact that the fermion kinetic terms cannot be affected by the unitary transformations, since they are proportional to the identity

in flavor space. The only terms we need to worry about are thus the vector-fermions interactions. Their general form was already computed in Eq. (7.78). Let us start with the neutral interactions involving A and Z . It is immediate to see that these interactions are of the form $\bar{f}_L \gamma^\mu f_L$ or $\bar{f}_R \gamma^\mu f_R$, i.e. being neutral they involve the same type of fermions. The unitary transformations we have performed cannot thus modify the form of the neutral interactions. The situation is different for the charged interactions. We have

$$\begin{aligned} \mathcal{L}_{charged} &= \frac{g}{\sqrt{2}} W_\mu^+ (\bar{Q} \gamma^\mu T^+ Q + \bar{L} \gamma^\mu T^+ L) + h.c. \\ &= \frac{g}{\sqrt{2}} W_\mu^+ (\bar{\nu}_L \gamma^\mu e_L + \bar{u}_L \gamma^\mu V_{CKM} d_L) + h.c. \end{aligned} \quad (7.102)$$

We discover that *charged interactions in the quark sectors are not flavor diagonal*. They are said to mediate tree-level flavor changing charged currents. We will come back on the flavor changing charged currents when discussing the experimental confirmations of the SM in Section 8.2.4. For the moment it is interesting to count the parameters in the CKM matrix. We will keep free the number of generations N because some important information can be extracted allowing for $N \neq 3$. At the end we will specialize to the $N = 3$ case. A generic $U(N)$ matrix has N^2 real parameters. Since $SO(N) \subset U(N)$, we identify $N(N-1)/2$ of the parameters with angles. We are thus left with $N^2 - N(N-1)/2 = N(N+1)/2$ parameters, which can only be phases. Are all these parameters real? The answer is no: fields redefinitions can be used to absorb some of the phases in the fields, since this transformation does not affect the physics. From Eq. (7.102) it is clear that we can transform u_L and d_L independently. However, only the relative phases matter, since transforming u_L and d_L in the same way does not affect the CKM matrix at all. We thus conclude that we can absorb $2N-1$ phases. This leaves $N(N+1)/2 - (2N-1) = (N^2 - 3N + 2)/2 = (N-1)(N-2)/2$ phases in the CKM matrices. The count of physical parameters is thus

$$\text{parameters} = \frac{N(N-1)}{2}(\text{angles}) + \frac{(N-1)(N-2)}{2}(\text{phases}). \quad (7.103)$$

If we lived in a world with only two generations then all the phases could be absorbed in the LH quark fields and the CKM would be completely real. Since we live in a world with 3 generations, though, the CKM matrix has $3(\text{angles}) + 1(\text{phase}) = 4$ parameters.

7.6 Consistency of the Standard Model

We have constructed the SM using a bottom up approach, introducing the minimum number of ingredients needed to correctly reproduce the experimental data. Our approach has been to construct a non-abelian gauge theory based on $SU(3)_c \times SU(2)_L \times$

$U(1)_Y$, giving mass to the appropriate gauge bosons using EWSB. Before considering EWSB, though, we need to couple the gauge bosons to conserved currents. Our argument has been completely classical, but as already noticed in Section 3.4 we need to make sure that no anomaly is present at the quantum level. The divergence of any current can be written as

$$\partial_\mu J_\mu^a = \left(\sum_{\text{LH}} A(R_L) - \sum_{\text{RH}} A(R_R) \right) \frac{d^{abc}}{128\pi^2} \epsilon^{\mu\nu\alpha\beta} (gF_{\mu\nu}^b)(gF_{\alpha\beta}^c), \quad (7.104)$$

where

$$A(R)d^{abc} = \text{tr} (T^a \{T^b, T^c\}) \quad (7.105)$$

is called ‘‘anomaly coefficient’’. The indices a, b and c can refer to different gauge groups.

We discuss in turn the various possibilities:

- $SU(3)_c^3$: the anomaly coefficient for $SU(3)_c^3$ is non trivial. However, we do not have to worry about its computation because QCD is a vector-like theory in which LH and RH fermions transform in the same way (both u_L and u_R are $SU(3)_c$ triplets for any generation, and the same is true for d_L and d_R). We thus conclude that $A(R_L) - A(R_R) = 0$ for any quark, and there is no anomaly;
- $SU(2)_L^3$: using $\{T_L^a, T_L^b\} = \{\tau^a, \tau^b\} = \delta^{ab}$ we see that the anomaly coefficient is proportional to $\text{tr}(\tau^a) = 0$ and hence it vanishes;
- $SU(3)_c SU(2)_L^2$: using again $\{T_L^a, T_L^b\} = \{\tau^a, \tau^b\} = \delta^{ab}$ and the fact that the Gell-Mann matrices are traceless we again conclude that this contribution vanishes;
- $SU(2)_L SU(3)_c^2$: in this case only Q contributes, since it is the only fermion charged under both $SU(2)_L$ and $SU(3)_c$. Since the space of weak and color charges mutually commute, the anomaly coefficient reduces to $\text{tr}(\tau^a)\text{tr}(\lambda^b\lambda^c) = 0$ and again vanishes. We can convince ourselves that this is the case also with an explicit computation. We explicitly exhibit both the $SU(2)_L$ and $SU(3)_c$ charges writing the quark doublet as $Q = (u_L^1, u_L^2, u_L^3, d_L^1, d_L^2, d_L^3)^T$ where the superscripts denote color. In this basis the symmetry generators are $T_c^a = \mathbf{1}_2 \otimes \lambda^a$ and $T_L^a = \tau^a \otimes \mathbf{1}_3$. The anomaly coefficient is thus equal to

$$\text{tr} (T_L^a \{T_c^b, T_c^c\}) = \text{tr} (\tau^a \otimes \{\lambda^b, \lambda^c\}) = \text{tr}(\tau^a)\text{tr}(\lambda^b\lambda^c) = 0; \quad (7.106)$$

- $SU(3)_c SU(2)_L^2$: analogously to what happens above, this anomaly coefficient also vanishes;
- $SU(3)_c U(1)_Y^2$ and $SU(2)_L U(1)^2$ and $SU(3)_c SU(2)_L U(1)_Y$: they all vanish following the argument above;
- $U(1)_Y SU(2)_L^2$ and $U(1) SU(3)_c^2$: in this case the anticommutator is non trivial, and we need to guarantee that $\text{tr}(Y_{f_L} - Y_{f_R}) = 0$ to cancel the anomaly. This is

the first non trivial constraint we obtain. Explicitly we have

$$\begin{aligned}\mathrm{tr}(Y_{f_L}) &= 3(\mathrm{gen}'\mathrm{s}) \times \left[3(\mathrm{color}) \times \left(\frac{1}{6} + \frac{1}{6} \right) + \left(-\frac{1}{2} - \frac{1}{2} \right) \right] = 0, \\ \mathrm{tr}(Y_{f_R}) &= 3(\mathrm{gen}'\mathrm{s}) \times \left[3(\mathrm{color}) \times \left(\frac{2}{3} - \frac{1}{3} \right) + (-1) \right] = 0.\end{aligned}\tag{7.107}$$

The anomaly coefficient vanishes. We notice that this is also the anomaly coefficient that enters the computations of the gravity-gravity- $U(1)_Y$ anomaly, which thus also vanishes;

- $U(1)_Y^3$: the last anomaly coefficient that must be checked is the pure abelian one involving the hypercharge only. We have

$$\begin{aligned}\mathrm{tr}(Y_{f_L}^3) &= 3(\mathrm{gen}'\mathrm{s}) \times \left[3(\mathrm{color}) \times \left(\left[\frac{1}{6} \right]^3 + \left[\frac{1}{6} \right]^3 \right) + \left(\left[-\frac{1}{2} \right]^3 + \left[-\frac{1}{2} \right]^3 \right) \right] = -\frac{2}{3}, \\ \mathrm{tr}(Y_{f_R}^3) &= 3(\mathrm{gen}'\mathrm{s}) \times \left[3(\mathrm{color}) \times \left(\left[\frac{2}{3} \right]^3 + \left[-\frac{1}{3} \right]^3 \right) + (-1)^3 \right] = -\frac{2}{3}.\end{aligned}\tag{7.108}$$

Again the total anomaly coefficient vanishes.

- Gravity: the only possible non-trivial anomaly coefficient is the hypercharge one.

We have

$$\begin{aligned}\mathrm{tr}(Y_{f_L}) &= 3(\mathrm{gen}'\mathrm{s}) \times \left[3(\mathrm{color}) \times \left(\frac{1}{6} + \frac{1}{6} \right) + \left(-\frac{1}{2} + -\frac{1}{2} \right) \right] = 0, \\ \mathrm{tr}(Y_{f_R}) &= 3(\mathrm{gen}'\mathrm{s}) \times \left[3(\mathrm{color}) \times \left(\frac{2}{3} - \frac{1}{3} \right) + (-1) \right] = 0.\end{aligned}\tag{7.109}$$

The SM can thus also be coupled consistently with gravity.

We thus conclude that the SM is a consistent gauge theory also at quantum level, being free from anomalies. It is important to notice that anomaly cancellation is guaranteed inside each generation. This is a consequence of lepton and quark universality.

7.7 Exact and approximate global symmetries of the SM

We conclude this chapter analyzing the symmetries of the SM Lagrangian. As we are going to see, some of these symmetries will be exact, others will only be approximate. We will not bother writing down the global symmetries associated with the gauge transformations, since they are symmetries of the Lagrangian by construction.

7.7.1 Symmetries of the kinetic terms: flavor symmetry and lepton/baryon number

While no additional symmetry is present in the Higgs kinetic term, there is a non-trivial flavor symmetry in the fermion kinetic Lagrangian. As we already observed, as long as we do not worry about the Yukawa terms, we are free to perform a unitary transformation on any fermion without changing the form of the kinetic term. This means that there is a global symmetry

$$U(3)_Q \times U(3)_u \times U(3)_d \times U(3)_L \times U(3)_e \equiv U(3)^5 \quad (7.110)$$

acting on the flavor indices of each fermion term. This symmetry is called *flavor symmetry*. Although it is a symmetry of the kinetic part of the fermion Lagrangian, this symmetry is explicitly broken by the Yukawa terms

$$-\mathcal{L}_{Yuk} = \bar{\mathbf{L}}Y_e\mathbf{e}_RH + \bar{\mathbf{Q}}Y_d\mathbf{d}_RH + \bar{\mathbf{Q}}Y_u\mathbf{u}_R\tilde{H}. \quad (7.111)$$

As we know from Sec. 7.5, the Yukawa matrix in the lepton sector can be completely diagonalized by a bi-unitary transformation, while the same is not true for the quark sector, in which only one of the two Yukawa matrices can be diagonalized. This means that we have a $U(1)^3$ symmetry in the lepton sector


$$(\mathbf{e}, \boldsymbol{\nu})_i \rightarrow e^{i\alpha_i}(\mathbf{e}, \boldsymbol{\nu}) \quad (7.112)$$

in which each generation transforms independently, and we have a $U(1)$ symmetry in the quark sector under which

$$(\mathbf{u}, \mathbf{d}) \rightarrow e^{i\frac{\alpha}{3}}(\mathbf{u}, \mathbf{d}). \quad (7.113)$$

The three $U(1)$ symmetries in the lepton sector are called *individual lepton numbers* (one for each of the electron, muon and tau flavors) while the $U(1)$ symmetry in the quark sector is called *baryon number*. The overall symmetry is thus

$$U(1)_B \times U(1)_e \times U(1)_\mu \times U(1)_\tau. \quad (7.114)$$

 **Exercise 7.4** Check that baryon number and the individual lepton numbers are anomalous. What are the non-anomalous combinations that can be taken without considering gravitational anomalies? How does the inclusion of gravitational anomalies change the previous conclusion?

The symmetries just described can be used to count the number of physical parameters coming from the Yukawa sector. The counting can be performed as follows: we first

identity the total number of parameters. We then observe that the Yukawa matrices explicitly break the flavor symmetries in such a way that only the broken generators can affect such Yukawa couplings (after all, an unbroken transformation is a symmetry of the Yukawa term and, by definition, cannot change it). The number of physical parameters is thus

$$\#\text{physical parameters} = \#\text{parameters} - \#\text{broken generators}. \quad (7.115)$$

Let us use this counting to deduce the number of physical parameters in the lepton and quark sectors:

- In the lepton sector we start with $(3 \times 3) \times 2 = 18$ real parameters (those of a 3×3 complex matrix). The initial symmetry is $U(3)_L \times U(3)_e$, explicitly broken to $U(1)^3$. The number of broken generators is thus $9 + 9 - 3 = 15$. We conclude that there are $18 - 15 = 3$ physical parameters in the lepton sector, given by the three charged leptons masses;
- In the quark sector we start with 36 parameters in the two Yukawa matrices Y_d and Y_u . The initial symmetry is $U(3)_Q \times U(3)_u \times U(3)_d$, explicitly broken to $U(1)_B$. The number of broken generators is thus $3 \times 9 - 1 = 26$, and the number of physical parameters is $36 - 26 = 10$. These are the six quark masses (3 for the up-type quarks and 3 for the down-type quarks), while the remaining 4 parameters are those appearing in the CKM matrix.

7.7.2 Discrete symmetries

Let us now analyze what happens to the discrete symmetries P (parity) and C (charge conjugation). Parity and Charge conjugation take a particularly simple form for fermions once we use the Weyl basis for the gamma matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad (7.116)$$

in which we have

$$\psi(t, \mathbf{x}) \xrightarrow{P} \gamma^0 \psi(t, -\mathbf{x}), \quad \psi(t, \mathbf{x}) \xrightarrow{C} -i\gamma^2 \psi(t, \mathbf{x})^*. \quad (7.117)$$

Under a CP transformation we thus have

$$\begin{aligned}
 \partial^\mu &\rightarrow \partial_\mu, \\
 \bar{\psi}_1 \psi_2 &\rightarrow \bar{\psi}_2 \psi_1, \\
 \bar{\psi}_1 \gamma_5 \psi_2 &\rightarrow \bar{\psi}_2 \gamma_5 \psi_1, \\
 \bar{\psi}_1 \gamma^\mu \psi_2 &\rightarrow -\bar{\psi}_2 \gamma_\mu \psi_1, \\
 \bar{\psi}_1 \gamma^\mu \gamma_5 \psi_2 &\rightarrow -\bar{\psi}_2 \gamma_\mu \gamma_5 \psi_1, \\
 V^\mu &\rightarrow -V_\mu^\dagger, \\
 h &\rightarrow h.
 \end{aligned} \tag{7.118}$$

 **Exercise 7.5** Show that the fermionic transformations above are correct.

What happens in the SM? It is easy to check that the kinetic terms, the mass terms, the interactions with the Higgs boson and the neutral currents are all CP invariant. The same applies to the charged interactions between leptons. The only term which is not manifestly invariant is the quark charged interactions, for which we have

$$\mathcal{L}_{CC} = \frac{g}{2\sqrt{2}} W_\mu^+ \bar{u} \gamma^\mu (1 - \gamma_5) V_{CKM} d + \frac{g}{2\sqrt{2}} W_\mu^- \bar{d} \gamma^\mu (1 - \gamma_5) V_{CKM}^\dagger u, \tag{7.119}$$

where we have already seen that the unitary CKM matrix has in general one complex phase. Applying a CP transformation we obtain

$$\mathcal{L}_{CC} \xrightarrow{CP} \frac{g}{2\sqrt{2}} W_\mu^- \bar{d} \gamma^\mu (1 - \gamma_5) V_{CKM} u + \frac{g}{2\sqrt{2}} W_\mu^+ \bar{u} \gamma^\mu (1 - \gamma_5) V_{CKM}^\dagger d. \tag{7.120}$$

The only way for CP to be respected is for the CKM matrix to be real, which in general will not be true. We conclude that *CP is explicitly violated in the SM with three generations.*

7.7.3 Chiral symmetry

As we already saw discussing mesons, there is an approximate chiral symmetry $SU(3)_L \times SU(3)_R$ in the QCD Lagrangian of the three light quarks, which is broken by the quark masses, by the EM interactions and by the weak interactions. We already saw the main utility of such symmetry in the construction of the Chiral Lagrangian in Chapter 6.2.

7.7.4 Custodial symmetry

Custodial symmetry is an approximate non-abelian symmetry whose origin can be traced back to the Higgs potential, but whose consequences can be seen already in the

Fermi Lagrangian. Let us first see how custodial symmetry manifest at low energy. As we saw in Section 7.1.2, we can derive Fermi theory from the SM looking at amplitudes in the $E \ll m_{W,Z}$ regime, when the propagator collapses to $1/m_{W,Z}^2$. We can obtain the same result proceeding in the following way: from the gauge part of the SM Lagrangian

$$\mathcal{L}_{gauge} = -\frac{1}{2}|W_{\mu\nu}^+|^2 - \frac{1}{4}(Z_{\mu\nu})^2 + m_W^2|W_\mu^+|^2 + \frac{m_Z^2}{2}(Z_\mu)^2 + \frac{g}{\sqrt{2}}(W_\mu^+ J_\mu^+ + h.c.) + \sqrt{g^2 + g'^2} Z_\mu J_\mu^Z \quad (7.121)$$

we can compute the equations of motions of the vectors bosons. We obtain

$$\begin{aligned} (\square + m_W^2) W_\mu^+ &= -\frac{g}{\sqrt{2}} J_\mu^-, \\ (\square + m_Z^2) Z_\mu &= -\sqrt{g^2 + g'^2} J_\mu^Z. \end{aligned} \quad (7.122)$$

Ignoring the kinetic terms (i.e. setting $\square \rightarrow 0$) we get the W and Z bosons as static fields, which can be completely eliminated from the Lagrangian by direct substitution:

$$\begin{aligned} \mathcal{L}_{gauge} &= m_W^2|W_\mu^+|^2 + \frac{m_Z^2}{2}(Z_\mu)^2 + \frac{g}{\sqrt{2}}(W_\mu^+ J_\mu^+ + h.c.) + \sqrt{g^2 + g'^2} Z_\mu J_\mu^Z \\ &= m_W^2 \left| \frac{-g J_\mu^-}{\sqrt{2} m_W^2} \right|^2 + \frac{m_Z^2}{2} \left[\frac{-\sqrt{g^2 + g'^2} J_\mu^Z}{m_Z^2} \right]^2 \\ &\quad + \frac{g}{\sqrt{2}} \left(\frac{-g J_\mu^-}{\sqrt{2} m_W^2} J_\mu^+ + h.c. \right) + \sqrt{g^2 + g'^2} \frac{-\sqrt{g^2 + g'^2} J_\mu^Z}{m_Z^2} \\ &= -\frac{g^2}{2m_W^2} J_\mu^+ J_\mu^- - \frac{g^2 + g'^2}{2m_Z^2} (J_\mu^Z)^2. \end{aligned} \quad (7.123)$$

This has to be compared with the Fermi theory enlarged to include neutral interactions,

$$\mathcal{L}_{Fermi} = -\frac{4G_F^{CC}}{\sqrt{2}} J_\mu^+ J_\mu^- - \frac{4G_F^{NC}}{\sqrt{2}} (J_\mu^Z)^2 = -\frac{4G_F^{CC}}{\sqrt{2}} \left[J_\mu^+ J_\mu^- + \frac{G_F^{NC}}{G_F^{CC}} (J_\mu^Z)^2 \right]. \quad (7.124)$$

We define the ρ parameter via

$$\rho = \frac{G_F^{NC}}{G_F^{CC}}, \quad (7.125)$$

i.e. the ρ parameter measures how much the charged current and neutral current interactions differ at low energy. Experimentally this parameter is close to 1. What is the prediction of the SM? Using the expressions for the vector boson masses in Eq. (7.81) we obtain

$$\rho = \frac{g^2 + g'^2}{m_Z^2} \frac{m_W^2}{g^2} = 1 \quad (\text{tree - level}). \quad (7.126)$$

Once radiative corrections are included the ρ parameter receives only small corrections. We thus suspect that there is an approximate symmetry protecting its value from being much different from unity. This symmetry is what is called *custodial symmetry*. The symmetry can be identified writing the charged currents as $J^\pm = J^1 \mp iJ^2$ and defining

$\Delta\rho = \rho - 1$ in the Fermi Lagrangian. We obtain

$$\mathcal{L}_{Fermi} = -\frac{4G_F^{CC}}{\sqrt{2}} [(J_\mu^1)^2 + (J_\mu^2)^2 + (J_\mu^Z)^2 + \Delta\rho(J_\mu^Z)^2] . \quad (7.127)$$

In the $\Delta\rho \rightarrow 0$ limit we see that the three currents J_μ^1 , J_μ^2 and J_μ^Z behave like a triplet of $SO(3)$. This is the custodial symmetry protecting the ρ parameter. Experimentally, $\Delta\rho \lesssim 0.01$.

Given the fact that the relation $\rho = 1$ is ultimately due to the gauge bosons masses, we suspect that the origin of custodial symmetry lies in the would-be NGBs, i.e. in the Higgs doublet sector. This suspicion is confirmed noticing that, as we are now going to show, the Higgs potential has an enhanced $SO(4)$ symmetry, spontaneously broken to the custodial $SO(3)$. To see this it is convenient to not work in unitary gauge, leaving the would-be NGB explicitly in the scalar sector:

$$H = \begin{pmatrix} \frac{\chi_1 + i\chi_2}{\sqrt{2}} \\ h + i\chi_3 \\ \frac{\chi_3}{\sqrt{2}} \end{pmatrix}, \quad (7.128)$$

where $\chi_{1,\dots,3}$ and h are the real components of the Higgs doublet. We now remember that the basic invariant appearing in the Higgs potential is $H^\dagger H = (\chi_1^2 + \chi_2^2 + \chi_3^2 + h^2)/2$. Last expression can be written as $H^\dagger H \equiv \phi^T \phi / 2$, where we have defined the real 4-component object $\phi = (\chi_1, \chi_2, \chi_3, h)^T = (\boldsymbol{\chi}, h)^T$. This means that the basic $SU(2)_L \times U(1)_Y$ invariant is actually invariant under a symmetry acting on ϕ as

$$\phi \rightarrow O \phi, \quad (7.129)$$

provided $O^T O = 1$. This is an $SO(4)$ symmetry, We conclude that, as promised, *the Higgs doublet scalar potential is invariant under an accidental $SO(4)$ symmetry*. What happens when we turn on the Higgs doublet vev? Using the $SO(4)$ symmetry we can always align the vev along the h direction, i.e.

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2}v \end{pmatrix}. \quad (7.130)$$

This vacuum is preserved by all those 4-dimensional rotations that act on the first three components only, i.e. by the $SO(3)$ subgroup of $SO(4)$. This is the custodial symmetry we were mentioning above. To make clearer the connection between what happens in the scalar sector and the properties of the gauge boson let us observe that $SO(4) \sim SU(2)_L \times SU(2)_R$, i.e. the generators of the $SO(4)$ Lie algebra can always be rearranged into two independent pairs of $SU(2)$ generators. For later convenience we will denote these $SU(2)$ groups with the subscripts L and R . They must not be confused

with the chiral symmetry. The Higgs doublet is more conveniently written as a 2×2 matrix

$$\Phi = \begin{pmatrix} \tilde{H} \\ H \end{pmatrix}, \quad (7.131)$$

where we remind the reader that $\tilde{H} = i\sigma_2 H^*$. The $SO(4) \sim SU(2)_L \times SU(2)_R$ transformation can be taken to be

$$\Phi \rightarrow U_L \Phi U_R^\dagger, \quad (7.132)$$

while the invariant appearing in the Higgs potential can be written as $H^\dagger H = \det \Phi$. The Higgs Lagrangian is thus

$$\mathcal{L}_H = \frac{1}{2} \langle D_\mu \Phi^\dagger D_\mu \Phi \rangle - (-\mu^2 \det \Phi + \lambda [\det \Phi]^2). \quad (7.133)$$

On the vacuum

$$\langle \Phi \rangle = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}, \quad (7.134)$$

and we see that this is left invariant by a diagonal transformation $U_L = U_R$, i.e. the diagonal subgroup $SU(2)_V$ can be identified with the custodial $SO(3)$. The matrix representation of the Higgs doublet is useful because it makes clear various facts:

- The $SU(2)_L$ transformation acts as the global version of the gauged $SU(2)_L$. This justifies its name;
- The $SU(2)_R$ group contains the global hypercharge transformations. It can be explicitly checked that they are generated by T_R^3 ;
- Given the observations above, we can write the Higgs covariant derivative as

$$D_\mu \Phi = \partial_\mu \Phi + ig W_\mu^a T_L^a \Phi - ig' B_\mu \Phi T_R^3. \quad (7.135)$$

Under a (global) $SU(2)_L$ transformation the W bosons transform as

$$W_\mu^a T_L^a \rightarrow U_L W_\mu^a T_L^a U_L^\dagger, \quad (7.136)$$

i.e. they are $SU(2)_L$ and $SU(2)_V$ triplets. This means that they transform precisely as the would-be NGBs χ introduced above. The $J_\mu^{1,2,3}$ currents, when promoted to custodial spurions, behave themselves as custodial triplets, thus justifying what we found in Eq. (7.127). We see immediately that \mathcal{L}_H is left invariant by this transformation. On the other hand, applying a (global) $SU(2)_R$ transformation we obtain

$$\Phi T_R^3 \rightarrow \Phi U_R^\dagger T_R^3 U_R \neq \Phi T_R^3 U_R^\dagger, \quad (7.137)$$

because a generic $SU(2)_R$ transformation does not commute with T_R^3 . The Lagrangian is thus non-invariant under this piece of the custodial transformation, meaning that *a non-vanishing g' explicitly breaks custodial symmetry.*


- Another source of explicit custodial breaking is given by the Yukawa couplings. Introducing the 2-components object $Q_R = (u_R, d_R)^T$ for each family, we can write the Yukawa Lagrangian as

$$\mathcal{L} = -\bar{Q}\Phi \begin{pmatrix} Y_u & 0 \\ 0 & Y_d \end{pmatrix} Q_R. \quad (7.138)$$


This term in the Lagrangian would be invariant under custodial symmetry only if $Y_u = Y_d$, a case which is clearly not compatible with our experimental observations, since it would imply the exact degeneracy of the up and down-type quarks of all generations. We conclude that *Yukawa couplings constitute another source of explicit custodial symmetry breaking*.

- Finally, we notice that the sources of explicit custodial breaking (g' and the Yukawa couplings) will contribute at loop level to $\Delta\rho$. The small experimental deviation from 1 can thus be explained as an effect of such breaking.


7.8 Additional exercises


 **Exercise 7.6** Consider an extension of the SM in which a scalar particle $T \sim (1, \mathbf{3}, 0)$ is added.


1. Enumerate all new invariants that can be added to \mathcal{L}_{SM} ;
2. What are the charges of the scalar fields contained in T ? Compute them explicitly;
3. Compute the scalar mass matrix supposing $\langle T \rangle = 0$;
4. Compute the scalar mass matrix supposing $\langle T \rangle \neq 0$;
5. Is it true that in the last case the neutrinos get a non-vanishing mass? Justify your answer;
6. Compute $\Gamma(T^0 \rightarrow \nu\bar{\nu})$, where T^0 is the neutral component in T .


 **Exercise 7.7** Suppose we extend the SM with an additional $U(1)_X$ symmetry, whose associated gauge boson will be called X_μ .


1. Is it true that we can write a term $B_{\mu\nu}X_{\mu\nu}$ in the Lagrangian? Why?
2. Is it true that we can write a term $W_{\mu\nu}^A X_{\mu\nu}$ in the Lagrangian? Why?
3. Suppose the Higgs boson is charged under the $U(1)_X$. What changes in the pattern of EWSB?

 **Exercise 7.8** Compute the cross section $e^+e^- \rightarrow \mu^+\mu^-$ in the SM as a function of the center-of-mass energy \sqrt{s} .

 **Exercise 7.9** Compute the annihilation cross section $e^+e^- \rightarrow W^+W^-$ in the SM as a function of the center-of-mass energy \sqrt{s} .

 **Exercise 7.10** Compute the annihilation cross section $e^+e^- \rightarrow e^+e^-$ in the SM as a function of the center-of-mass energy \sqrt{s} .

 **Exercise 7.11** We want to construct a theory based on the symmetry breaking pattern $SU(3)_1 \times SU(2)_2 \rightarrow SU(3)_c$, where $SU(3)_c$ is the usual color group. Which scalar representation would you add to the theory to achieve this symmetry breaking? Justify your answer.

 **Exercise 7.12** How can we include the effects of weak interactions in the chiral Lagrangian?

Chapter 8 Experimental confirmations of the Standard Model

In the previous chapter we have constructed the Standard Model, outlining some of its experimental consequences. In this chapter we look more thoroughly to the SM predictions and to data. It is worth emphasizing from the beginning that there is a spectacular agreement between the SM predictions and experimental results.

8.1 An overview of data

As we saw, the SM was proposed in the 1960's. At that time, the known particles were the u , d and s quarks, as well as the first two generations of leptons. During the 1970s various fundamental discoveries were made:

- experimental observation of neutral current processes at the Gargamelle experiment in 1973. The processes observed were $\nu e \rightarrow \nu e$, $\nu p \rightarrow \nu p$ and $\nu n \rightarrow \nu n$, exactly the processes predicted by the exchange of a Z boson;
- the third generation of leptons was discovered in 1975 via the first experimental detection of the τ charged lepton;
- the charm quark was discovered in 1974, completing the second generation of quarks;
- in 1978 the bottom quark was discovered, with the same quantum numbers of the down and strange quarks.

By the end of the 1970s it was clear that the SM had good chances of being a successful theory. The only missing elements were the top quark (to close the third generation of quarks), the W and Z bosons (although charged and neutral currents were indirect evidence of their existence) and the Higgs boson. The W and Z bosons were discovered in 1983 at the UA1 and UA2 experiments, completing the gauge structure of the SM. The top quark was discovered by the Fermilab-Tevatron accelerator (with $p\bar{p}$ collisions) in 1995, while the Higgs boson was discovered only in 2012 at the CERN-LHC (via pp collision).

Interestingly, many properties of the top quark and Higgs boson were known before their discovery thanks to the precision program undergone at the CERN-LEP experiments during the 1990s. The LEP-I and LEP-II accelerators collided e^+e^- at a center-of-mass (CM) energy of $\sqrt{s} \simeq m_Z$ (LEP-I) and up to $\sqrt{s} = 210$ GeV (LEP-II). Since the initial

state is leptonic, the environment is not “polluted” by too many QCD events, and the typical precision reached was of the order of ‰. As already mentioned, the agreement with the SM predictions was very good.

The LEP experimental program was too rich to present here all the results. We will thus content ourselves to give a taste of how the data-theory comparison is done. The idea is pretty simple: suppose the theory depends on N parameters. What we need to test the theory are $N' > N$ measurements, N of which will be used to create a 1 : 1 correspondence with the parameters. These are called *input parameters*. Once this is done, all the parameters are expressed in terms of observed quantities, and we can compute the observables corresponding to the remaining $N' - N$ measurements in terms of known quantities. If the predictions agree with the measurements the theory is valid; if not, we need to consider a different theory.

We present here a list of measurements involving the SM gauge sector:

1. electron magnetic moment

$$\frac{g_e}{2} = 1.0011 \dots \pm 2.8 \times 10^{-13} , \quad (8.1)$$

used to extract the electromagnetic fine structure constant α_e ;

2. Muon lifetime

$$\tau_\mu^{-1} = 2.99598 \times 10^{-19} \text{ GeV} , \quad (8.2)$$

used to extract the Fermi constant G_F ;

3. Z mass at the pole

$$m_Z = 91.1876 \pm 0.0021 \text{ GeV} ; \quad (8.3)$$

4. W mass at the pole

$$m_W = 80.385 \pm 0.015 \text{ GeV} ; \quad (8.4)$$

5. Polarization asymmetry

$$A_e = \frac{\sigma(e_L^- e_L^+ \rightarrow Z) - \sigma(e_R^- e_R^+ \rightarrow Z)}{\sigma(e_L^- e_L^+ \rightarrow Z) + \sigma(e_R^- e_R^+ \rightarrow Z)} = 0.1515 \pm 0.0019 ; \quad (8.5)$$

6. Total Z boson decay width

$$\Gamma_Z = 2.4952 \pm 0.0023 \text{ GeV} . \quad (8.6)$$

More observables have been measured, but we will not consider them here. Notice that we can distinguish between low energy measurements (electron magnetic moment and muon lifetime) and measurements performed at the Z -pole (all the others in the list).

8.2 Predictions at tree-level

We now turn to the predictions of the SM at tree-level. We discuss in turn predictions in the gauge and in the flavor sector.

8.2.1 Gauge sector

The parameters in the gauge sector of the SM are the gauge bosons masses and couplings. They can all be expressed in terms of the three parameters

$$\{g_0, g'_0, v_0\} \leftrightarrow \{e_0, s_0^2, v_0^2\}, \quad (8.7)$$

where s_0^2 is the square of the weak angle. We use the subscript 0 to remember that these are the parameters appearing in the Lagrangian, and must not be confused with the experimental observables. The choice of the second set of variables will make some computation simpler.¹

Let us follow the program outlined above: we need to choose three experimental quantities and express the parameters g , g' and v in terms of these quantities. Once this is done, we use the SM to make predictions about the other experimentally measured observables and compare with data. We will choose the three better measured quantities as input parameters: g_e (from which we will extract $\alpha_e(0)$ at zero energy), G_F and m_Z , in such a way that we need to connect

$$\{e_0, s_0^2, v_0\} \leftrightarrow \{\alpha_e, G_F, m_Z\}. \quad (8.8)$$

8.2.1.1 Extracting the parameters 1: α_e from the electron magnetic moment

The computation of the electron magnetic moment was one of the first QFT computations to be found compatible with experiments (Schwinger, 1948). The result is now known in QED at 5-loop. For our purposes we just consider the 1-loop result

$$g_e - 2 = \frac{\alpha_e(0)}{\pi} = \frac{e_0^2}{4\pi^2}. \quad (8.9)$$

Using the value of the electron magnetic moment given in Eq. (8.1) we obtain

$$\alpha_e(0) = (137.03 \dots \pm 4.4 \cdot 10^{-8})^{-1}. \quad (8.10)$$

¹There is a caveat: using s_0^2 as parameter we are implicitly assuming lepton universality. While this is a feature of the SM, it is important to be aware that in more generic models this feature may not be present, and s_0^2 will not be a good variable.

²Notice that since we are not considering loop corrections, the electric coupling is not running and we do not need to worry about the scale at which it is computed.

8.2.1.2 Extracting the parameters 2: G_F from the muon decay width

As already said, the Fermi constant G_F is extracted from the muon decay width. More specifically, we look at the $\mu \rightarrow e\nu_\mu\bar{\nu}_e$ decay, which constitutes basically 100% of the branching ratio. Given the importance of this process, let us compute explicitly $\Gamma(\mu \rightarrow e\nu_\mu\bar{\nu}_e)$ in the Fermi theory. The amplitude for the process $\mu(p_0) \rightarrow \nu_\mu(p_1) + e(p_2) + \bar{\nu}_e(p_3)$ is

$$i\mathcal{A} = \frac{-4iG_F}{\sqrt{2}} (\bar{u}_1\gamma^\mu P_L u_0) (\bar{u}_2\gamma_\mu P_L v_3) . \quad (8.11)$$

The averaged squared amplitude thus results in

$$\begin{aligned} |\overline{\mathcal{A}}|^2 &= \frac{1}{2} \frac{16G_F^2}{2} \text{tr} (\bar{u}_1\gamma^\mu P_L u_0 \bar{u}_0\gamma^\alpha P_L u_1) \text{tr} (\bar{u}_2\gamma_\mu P_L v_3 \bar{v}_3\gamma_\alpha P_L u_2) \\ &= 4G_F^2 \text{tr} (\not{p}_1\gamma^\mu P_L (\not{p}_0 + m_\mu)\gamma^\alpha P_L) \text{tr} ((\not{p}_2 + m_e)\gamma_\mu P_L \not{p}_3\gamma_\alpha P_L) \\ &= 64G_F^2 (p_0 \cdot p_3)(p_1 \cdot p_2). \end{aligned} \quad (8.12)$$

The main difficulty is now to integrate over the three-body phase space. Following the Particle Data Group prescription (see the documents “Kinematics”) we have ³

$$d\Gamma = \frac{1}{(2\pi)^3} \frac{1}{32m_\mu^3} |\overline{\mathcal{A}}|^2 dm_{12}^2 dm_{23}^2 , \quad (8.13)$$

where $m_{ij}^2 = (p_i + p_j)^2$. For a fixed value of m_{12}^2 the m_{23}^2 integration is in the interval $(e_2 + e_3)^2 - (\sqrt{e_2^2 - m_2^2} + \sqrt{e_3^2 - m_3^2})^2 \leq m_{23}^2 \leq (e_2 + e_3)^2 - (\sqrt{e_2^2 - m_2^2} - \sqrt{e_3^2 - m_3^2})^2$,

where

$$e_2 = \frac{m_{12}^2 - m_1^2 + m_2^2}{2m_{12}}, \quad e_3 = \frac{M^2 - m_{12}^2 - m_3^2}{2m_{12}} \quad (8.15)$$

in terms of the masses m_i of the final states and of the mass M of the initial state. The m_{12}^2 integration instead in in the interval

$$(m_1 + m_2)^2 \leq m_{12}^2 \leq (M - m_3)^2 . \quad (8.16)$$

The previous equations follow from the analysis of the kinematics endpoints. In our case we obtain

$$\Gamma(\mu \rightarrow \nu_\mu e \bar{\nu}_e) = \frac{G_F^2}{192\pi^3} \left(m_\mu^5 - 8m_e^2 m_\mu^3 + \frac{8m_e^6}{m_\mu} - \frac{m_e^8}{m_\mu^3} + 12m_e^4 m_\mu \log \frac{m_\mu^2}{m_e^2} \right) . \quad (8.17)$$

 **Exercise 8.1** Derive the result above.

³Another reference in which explicit formulas for multi-particle phase spaces can be found is 1210.7939.

Since the electron and muon masses are well measured, we can use the experimental value of τ_μ to extract

$$G_F = 1.16393 \times 10^{-5} \text{ GeV}^{-2}. \quad (8.18)$$

Notice that at tree level in the SM we have

$$G_F = \frac{\sqrt{2}g_0^2}{8m_{W0}^2} = \frac{\sqrt{2}g_0^2}{8\frac{g_0^2 v_0^2}{2}} = \frac{1}{2\sqrt{2}v_0^2}, \quad (8.19)$$

from which we obtain

$$v_0 \simeq 174 \text{ GeV}. \quad (8.20)$$

8.2.2 Extracting the parameters 3: Z mass

At tree level in the SM the Z mass is the easiest of the input parameters, since we simply have

$$m_Z^2 = \frac{g_0^2 + g_0'^2}{2} v_0^2 = \frac{e_0^2}{2s_0^2(1-s_0^2)} v_0^2. \quad (8.21)$$

8.2.3 Putting all together: the SM gauge sector at tree-level

Let us now put everything together and express the SM parameters $\{g, g', v\}$ in terms of the observables $\{\alpha_e(0), G_F, m_Z\}$ at tree level. From the relations

$$\alpha_e(0) = \frac{e_0^2}{4\pi}, \quad G_F = \frac{1}{2\sqrt{2}v_0^2}, \quad m_Z^2 = \frac{e_0^2}{2s_0^2(1-s_0^2)} v_0^2 \quad (8.22)$$

we obtain

$$\begin{aligned} e_0 &= \sqrt{4\pi\alpha_e(0)} \simeq 0.312224 \dots \pm 9 \times 10^{-7}, \\ s_0^2 &= \frac{1}{2} \left(1 - \frac{\sqrt{G_F m_Z^2 (G_F m_Z^2 - \sqrt{4\pi\alpha_e(0)})}}{G_F m_Z^2} \right) \simeq 0.23178(1), \\ v_0 &= \frac{1}{2^{3/4} \sqrt{G_F}} \simeq 174.287(1) \text{ GeV}. \end{aligned} \quad (8.23)$$

We have estimated the errors with a simple quadrature propagation to have an idea of their size. These values allow us to make predictions for other observables at tree level in the SM. For instance, the W mass becomes

$$m_W = \frac{g_0}{\sqrt{2}} v_0 = \frac{e_0}{2s_0} v_0 = 80.907(3) \text{ GeV}. \quad (8.24)$$

We see that our prediction is not in agreement with data (as a matter of fact, we are roughly 35σ away from the measured value). The same is true for other observables. We thus conclude that *the SM predictions at tree-level are not compatible with experiments*. As we are going to see in Section 8.3, loop corrections will make the SM predictions compatible with data, marking one of the greatest successes of Quantum Field Theory.

8.2.4 Flavor sector

Before turning to the loop corrections in the gauge sector, let us comment on the experimental confirmations in the flavor sector. As we saw in the previous chapter one of the main phenomenological predictions of the SM is that charged current interactions are mediated by the CKM matrix V_{CKM}

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} W_\mu^+ \bar{\mathbf{u}}_L \gamma^\mu V_{CKM} \mathbf{d}_L + h.c. \quad (8.25)$$

The SM prediction is that such matrix is unitary. The unitarity condition can be expressed as

$$\sum_i V_{ik}^* V_{ij} = \delta_{jk} \Rightarrow V_{1k}^* V_{1j} + V_{2k}^* V_{2j} + V_{3k}^* V_{3j} = 0 \quad (8.26)$$

for $j \neq k$. This is the sum of three complex numbers, i.e. a triangle in the complex plane. To check the SM predictions we need to check the closure of one of such triangles in the complex plane. To understand which choice of k and j flavor indices is more convenient we need some information about the structure of the CKM matrix. A useful phenomenological parametrization of the CKM matrix is due to Wolfenstein (1983)

$$V_{CKM}^W = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda - iA\lambda^5\eta & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 - iA\eta\lambda^4 & 1 \end{pmatrix}, \quad (8.27)$$

where $\lambda \simeq 0.23$ is the Cabibbo angle. This matrix is unitary until $\mathcal{O}(\lambda^4)$. Since λ is small, we see that the third generation of quarks does not interact much with the first two.

The standard choice for the *unitary triangle* is $k = d$ and $j = b$, i.e.

$$\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} + 1 = 0. \quad (8.28)$$

The experimental program aiming to measure the CKM matrix elements to check the closure of the unitary triangle is extremely reach, and would deserve a series of separate lectures.⁴ Here we just limit ourselves to give an example of how the sides of the triangle can be measured.


Let us focus on V_{ud} which, according to the Wolfenstein parametrization, gives us a measurement of the Cabibbo angle λ . The key process is the $u \rightarrow d$ transition. What are possible physical processes in which this transition happens? We can list (i) Nuclear β -decay (for instance ${}^3\text{H} \rightarrow {}^3\text{He}$ or ${}^{14}\text{C} \rightarrow {}^{14}\text{N}$), (ii) β -decay in which a free neutron decays in a free proton, and (iii) pion decays $\pi^+ \rightarrow \pi^0 e \nu$ or $\pi^+ \rightarrow \mu \nu$. At the level of

⁴See for instance arXiv:1501.05283 and arXiv:1711.03624 for some relatively recent lecture sets of good quality.

Fermi Lagrangian the operators involved are always the same. What changes, however, are the initial and final states involved in the hadronic part. To compute the amplitudes for the decay we thus need to know the matrix elements

$$\langle \mathcal{N}' | \bar{u}_L \gamma^\mu d_L | \mathcal{N} \rangle, \quad \langle p | \bar{u}_L \gamma^\mu d_L | n \rangle, \quad \langle \pi^0 | \bar{u}_L \gamma^\mu d_L | \pi^+ \rangle, \quad \langle 0 | \bar{u}_L \gamma^\mu d_L | \pi^+ \rangle. \quad (8.29)$$

Such matrix elements cannot be computed from first principles, but must be extracted from experiments. Their form can be inferred using Lorentz and CP covariance, as well as $SU(3)_V$ selection rules (see arXiv:1711.03624 for a nice discussion of these points). In the case of the pions we can also use Chiral Perturbation Theory with the inclusion of the interactions with leptons, but this approach clearly cannot work for the nucleons or, even worse, for the nuclei involved in the β -decay.

 **Exercise 8.2** Using the techniques we introduced in Chapter 6.3 compute the decay width $\Gamma(\pi^+ \rightarrow \pi^0 e \bar{\nu})$. *Hint:* it is convenient to introduce that coupling with the W boson using the same techniques used to include the coupling with photons.

8.3 Predictions at loop level and the triumph of the Standard Model

We are finally ready to discuss the loop predictions of the SM and compare it with data. Given the complexity of the subject we will not attempt a complete treatment. We will instead simply consider the so-called *oblique (or universal) corrections*, analogous to the universal corrections considered in our discussion of QED radiative corrections in Sec. 3.5. It is important to stress from the very beginning a technical complication: while in QED the oblique corrections, computed as corrections to the photon propagator, are gauge invariant, this is not so for non-abelian gauge theories (in particular the SM). To recover gauge invariance some universal contribution to the vertex function must be included in the self energies, making the computation not so straightforward. We will not perform the complete computation, simply referring the interested reader to the clear treatments of *Nucl.Phys.B 322 (1989) 1-54* and *Nucl.Phys.B 350 (1991) 25-72*.

The general idea is the same we used in Sec. 8.2.1: (i) we identify the input parameters, (ii) we compute them in terms of the parameters of the Lagrangian, (iii) we invert the relations in such a way that the parameters of the Lagrangian are expressed in terms of the observables, and (iv) we predict other quantities in terms of the input

parameters to compare with experiments. In Sec. 8.2.1 we traded $\{\alpha_e(0), G_F, m_Z\} \leftrightarrow \{e_0, s_0^2, v_0\}$. It will now be convenient to trade $\{\alpha_e(0), G_F, m_Z\} \leftrightarrow \{e_0^2, s_0^2, m_{Z0}^2\}$.

When considering loop corrections we need to carefully identify how to express the observables in terms of the parameters. In the case of the SM three difficulties arise:

- The vector vacuum polarization is of the form

$$\Pi^{\mu\nu} = \Pi_T(q^2)g^{\mu\nu} + \Pi_L(q^2)q^\mu q^\nu . \quad (8.30)$$

We have already seen that, in the case of the photon, only the $\Pi_T(q^2)$ term is relevant, since the other vanishes when contracted with the conserved fermion current. The same reasoning cannot be applied to massive vectors: as we saw, the massive vectors in the SM couple to currents that are non-conserved even at the classical level. This is due to the fact that the interactions are chiral, inducing an axial term $J_5^\mu = \bar{f}\gamma^\mu\gamma_5 f$ in the current;

- Loop corrections generate a vacuum polarization Π_{AZ} (for instance, imagine a fermion loop with a photon on one side and a Z boson on the other);
- Not all the vacuum polarizations are gauge invariant. The difficulty arises because of the non-abelian nature of the W and Z interactions, and is not present in QED.

We will not give a complete treatment of all these difficulties, since it would be more technical than the level of these lectures. Let us simply outline the solutions of the various problems:

- The longitudinal part of the vector polarization is irrelevant also for massive vectors. It is easy to show that, when the fermions are on-shell, the divergence of this term is proportional to the fermion mass. In momentum space this implies $q_\mu J_5^\mu \propto m_f$. Since the fermions we will be interested in are very light, this term can be safely neglected in all computations. This means that even for massive vectors the only term that will matter is the transverse correction to the propagator, $\Pi_T(q^2)$. To simplify the notation, we will drop the subscript T , leaving implicit that the only vacuum polarization correction that matter is the transverse one;
- To eliminate the mixing generated by Π_{AZ} we will diagonalize again the neutral vector sector, to identify the correct physical states. We will perform this diagonalization explicitly below;
- To guarantee a gauge invariant result we need to consider also the universal corrections of the trilinear vertex functions. Once this is done, all the vector polarizations are completely gauge invariant. To make this clear, we introduce the notation

$$\Pi_{XY} = \Pi_{XY}(0) + q^2\Pi'_{XY}(q^2) \quad (8.31)$$

(notice that, although the notation suggests otherwise, the second term coincides with the derivative of Π_{XY} only in $q^2 = 0$. We use this notation to conform to what can usually be found in the literature). In particular, gauge invariance implies that all the vacuum polarizations associated with photons are proportional to q^2 as in QED:

$$\Pi_{AA} = q^2 \Pi'_{AA}(q^2), \quad \Pi_{AZ} = q^2 \Pi'_{AZ}(q^2), \quad (8.32)$$

with both Π'_{AA} and Π'_{AZ} finite in the $q^2 \rightarrow 0$ limit.

 **Exercise 8.3** Show that $\partial_\mu J_5^\mu \propto m_f$.

Let us now diagonalize explicitly the neutral vector sector taking into account radiative corrections. The diagonalization is more transparent writing the Lagrangian in momentum space. To do so, we express the fields as

$$A(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} A(p) = \int \frac{d^4 q}{(2\pi)^4} e^{iqx} A(-p), \quad (8.33)$$

where the choice between $A(p)$ and $A(-p)$ is dictated by the need of simplifying the expression in such a way that

$$\int d^4 x e^{i(p-q)x} = (2\pi)^4 \delta(p-q) \quad (8.34)$$

appears. For instance, the action of a real scalar field in momentum space reads

$$\begin{aligned} S &= \int d^4 x \left(\frac{1}{2} \partial_\mu \phi \partial_\mu \phi - \frac{m^2}{2} \phi^2 \right) \\ &= \int d^4 x \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{2} e^{-i(p-q)x} \phi(-q) (pq - m^2) \phi(p) \\ &= \int \frac{d^4 p}{(2\pi)^4} \frac{1}{2} \phi(-q) (p^2 - m^2) \phi(p) \\ &= \int \frac{d^4 p}{(2\pi)^4} \mathcal{L}_p. \end{aligned} \quad (8.35)$$

We see that with the action written in momentum space we can read immediately the inverse of the propagator. Applying the same procedure to the vector boson Lagrangian of the SM including radiative corrections we obtain

$$\begin{aligned} \mathcal{L}_q &= -\frac{1}{2} q^2 A^2 + \frac{1}{2} q^2 \Pi'_{AA} A^2 - \frac{1}{2} (q^2 - m_{Z0}^2 - \Pi_{ZZ}) Z^2 \\ &\quad + q^2 \Pi'_{AZ} AZ + e_0 A J_{EM} + \frac{e_0}{s_0 c_0} Z J_Z \\ &\quad - (q^2 - m_{W0}^2 - \Pi_{WW}) W^+ W^- + \frac{g_0}{\sqrt{2}} W^+ J^+ + h.c. \end{aligned} \quad (8.36)$$

where we have suppressed all Lorentz indices. The Z current can be written as (see

Eq. (7.78))

$$J_Z = J_{3L} - s_0^2 J_{EM}. \quad (8.37)$$

We can see clearly that there is a mixing (which is called *kinetic*, since it depends on q^2) between the photon and the Z bosons. As with mass mixing, also kinetic mixing signals the fact that we are still not dealing with physical states (we do not expect a physical state to change into another particle while propagating). We thus need to diagonalize the kinetic terms. To do so we proceed as follows:


- We first redefine

$$A \rightarrow \frac{A}{\sqrt{1 - \Pi'_{AA}}} \quad (8.38)$$

to make the kinetic term of the photon canonical (i.e. simply proportional to q^2 without any other momentum dependence);

- Once this is done, we shift $A \rightarrow A + \xi Z$ with ξ chosen in such a way that the mixing term proportional to Π'_{AZ} is eliminated from the Lagrangian. It is immediate to verify that we must choose

$$\xi = \frac{\Pi'_{AZ}}{\sqrt{1 - \Pi'_{AA}}}. \quad (8.39)$$

 **Exercise 8.4** Compute explicitly ξ and verify the claim above.

After these manipulation we obtain

$$\begin{aligned} \mathcal{L}_q = & -\frac{1}{2}q^2 A^2 - \frac{1}{2} \left[q^2 \left(1 - \frac{(\Pi'_{AZ})^2}{1 - \Pi'_{AA}} \right) - m_{Z0}^2 - \Pi_{ZZ}(q^2) \right] Z^2 \\ & + \frac{e_0}{\sqrt{1 - \Pi'_{AA}}} A_\mu J_{EM}^\mu + \frac{e_0}{s_0 c_0} Z J_{3L} - \frac{e_0}{s_0 c_0} \left(s_0^2 - s_0 c_0 \frac{\Pi'_{AZ}}{1 - \Pi'_{AA}} \right) Z J_{EM} \\ & - (q^2 - m_{W0}^2 - \Pi_{WW}) W^+ W^- + \frac{g_0}{\sqrt{2}} W^+ J^+ + h.c. \end{aligned} \quad (8.40)$$

We are now in a position to extract the physical consequences from the Lagrangian Eq. (8.40). Since we expect the vacuum polarizations to be (at least) 1-loop suppressed, we will work to first order in the Π 's. First of all, we see that we recover the running electric charge

$$e^2(q^2) = \frac{e_0^2}{1 - \Pi'_{AA}(q^2)}. \quad (8.41)$$

We obtain the same result as in QED. Let us now turn to the operative definition of the Fermi constant. We can define it as the zero energy limit of the charged current amplitude. A small difficulty arises, however: since the W boson is unstable, the vacuum polarization can develop a non-vanishing imaginary part which, according to

the Optical Theorem (see Appendix D), can be written as

$$\text{Im}[\Pi_{WW}(q^2)] = \sqrt{s} \Gamma_W, \quad (8.42)$$

where $s = q^2$ and Γ_W the total W decay width. An immediate consequence of this fact is that we need to worry only about the real part of Π_{WW} in our computations. It is useful to define the *physical (or pole) mass* as the value of the pole of the real part of the propagator,

$$m_W^2 - m_{W0}^2 - \text{Re}[\Pi_{WW}(m_W^2)] = 0. \quad (8.43)$$

Going back to the definition of the Fermi constant, we obtain

$$\begin{aligned} \mathcal{M}_{CC} &= -\frac{4G_F}{\sqrt{2}} J^+ J^- = \lim_{q^2 \rightarrow 0} \frac{e_0^2}{2s_0^2} \frac{1}{q^2 - m_{W0}^2 - \text{Re}[\Pi_{WW}(q^2)] - i\sqrt{s}\Gamma_W} J^+ J^- \\ &= -\frac{e_0^2}{2s_0^2} \frac{1}{(1 - s_0^2)m_{Z0}^2 + \text{Re}[\Pi_{WW}(0)]} J^+ J^-, \end{aligned} \quad (8.44)$$

from which we immediately conclude

$$\frac{4G_F}{\sqrt{2}} = \frac{e_0^2}{2s_0^2} \frac{1}{(1 - s_0^2)m_{Z0}^2 + \text{Re}[\Pi_{WW}(0)]}. \quad (8.45)$$

We finally turn to the operative definition of the Z boson mass. The situation is conceptually similar to that of the W boson. What is measured in the experiments is the pole mass, defined to be the zero of the inverse propagator:

$$\begin{aligned} m_Z^2 \left(1 - \frac{[\Pi'_{AZ}(m_Z^2)]^2}{1 - \Pi'_{AA}(m_Z^2)} \right) - m_{Z0}^2 - \Pi_{ZZ}(m_Z^2) &= 0 \\ m_Z^2 - \Pi_{ZZ}(m_Z^2) - m_{Z0}^2 &\simeq 0. \end{aligned} \quad (8.46)$$

We now have all the ingredients we need to carry on the precision program: We were able to find three expressions connecting the measured parameters $\{\alpha_e(0), G_F, m_Z\}$ to the bare parameters $\{e_0^2, m_{Z0}^2, s_0^2\}$. The relations can be inverted to express the bare parameters in terms of the inputs. To simplify the expressions, it is convenient to define the tree-level value of the squared of the weak angle,

$$s_{tree}^2 = \frac{1}{2} - \frac{1}{2} \left(1 - \frac{e^2(0)}{\sqrt{2}G_F m_Z^2} \right)^{1/2}. \quad (8.47)$$

We obtain

$$\begin{aligned} e_0^2 &\simeq e^2(0) \left(1 - \Pi'_{AA}(0) \right), \\ m_{Z0}^2 &\simeq m_Z^2 - \text{Re}[\Pi_{ZZ}(m_Z^2)], \\ s_0^2 &\simeq s_{tree}^2 \left[1 - \frac{c_{tree}^2}{c_{tree}^2 - s_{tree}^2} \Pi_R \right], \end{aligned} \quad (8.48)$$

with

$$\Pi_R = \frac{\text{Re}[\Pi_{WW}(0)]}{c_{tree}^2 m_Z^2} - \frac{\text{Re}[\Pi_{ZZ}(m_Z^2)]}{m_Z^2} + \Pi'_{AA}(0). \quad (8.49)$$

We are now in the position to make predictions. We focus on the pole W mass

$$\begin{aligned}
 m_W^2 &= m_{W0}^2 + \text{Re}[\Pi_{WW}(m_W^2)] = (1 - s_0^2)m_{Z0}^2 + \text{Re}[\Pi_{WW}(m_W^2)] \\
 &= c_{tree}^2 m_Z^2 + \text{Re}[\Pi_{WW}(m_W^2)] - c_{tree}^2 \text{Re}[\Pi_{ZZ}(m_Z^2)] + \frac{s_{tree}^2 c_{tree}^2}{c_{tree}^2 - s_{tree}^2} \Pi_R \\
 &= c_{tree}^2 m_Z^2 \left(1 + \frac{\text{Re}[\Pi_{WW}(m_W^2)]}{c_{tree}^2 m_Z^2} - \frac{\text{Re}[\Pi_{ZZ}(m_Z^2)]}{m_Z^2} + \frac{s_{tree}^2}{c_{tree}^2 - s_{tree}^2} \Pi_R \right).
 \end{aligned} \tag{8.50}$$

The explicit computation of the vacuum polarizations can be done using the results in Appendix E.

Exercise 8.5

- Compute explicitly the vacuum polarizations entering in the pole W mass;
- Verify that all the divergences cancel out (pay particular attention to the role played by the CKM matrix).

Once radiative corrections are included the SM prediction of the pole W mass is

$$m_W = 80.368 \text{ GeV}, \tag{8.51}$$

now in good agreement with experiments. This is but an example of the remarkable success of the SM in predicting observables.

8.4 Electroweak parameters

The procedure we have described in the previous section can be employed even if additional particles not present in the SM are considered. As a matter of fact, the precision tests of the SM and its success are a powerful tool in setting bounds on possible new states. To contain our discussion to a well motivated example, let us introduce the so-called “electroweak (or precision) parameters” S , T and U . Our starting point are the vacuum polarizations in Eq. (8.31). It will be convenient to consider such vacuum polarizations in the original $(W_\mu^3, B_\mu, W_\mu^+, W_\mu^-)$ basis, rather than in the mass eigenbasis.:

$$\begin{aligned}
 \Pi_{33}(q^2) &= \Pi_{33}(0) + q^2 \Pi'_{33}(q^2), & \Pi_{BB}(q^2) &= \Pi_{BB}(0) + q^2 \Pi'_{BB}(q^2), \\
 \Pi_{3B}(q^2) &= \Pi_{3B}(0) + q^2 \Pi'_{3B}(q^2), & \Pi_{WW}(q^2) &= \Pi_{WW}(0) + q^2 \Pi'_{WW}(q^2).
 \end{aligned} \tag{8.52}$$

In terms of the vacuum polarizations in the physical basis, we can write

$$\begin{aligned}\Pi_{33} &= s_0^2 \Pi_{AA} + c_0^2 \Pi_{ZZ} + 2s_0 c_0 \Pi_{AZ}, \\ \Pi_{BB} &= c_0^2 \Pi_{AA} + s_0^2 \Pi_{ZZ} - 2s_0 c_0 \Pi_{AZ}, \\ \Pi_{3B} &= s_0 c_0 (\Pi_{AA} - \Pi_{ZZ}) + (c_0^2 - s_0^2) \Pi_{AZ}.\end{aligned}\tag{8.53}$$

Let us now suppose that the new physics states are heavier than the SM particles ⁵, in such a way that $q^2 \ll \Lambda^2$. In this regime it is safe to set $q^2 = 0$ in the Π'_{XY} form factors. This leaves us with 8 form factors: $\{\Pi_{WW}(0), \Pi_{33}(0), \Pi_{33}(0), \Pi_{3B}(0), \Pi'_{WW}(0), \Pi'_{33}(0), \Pi'_{33}(0), \Pi'_{3B}(0)\}$.

Not all of them are free: gauge invariance requires the two constraints

$$\Pi_{ZZ}(0) = \frac{\Pi_{33}(0)}{c_0^2} = \frac{\Pi_{BB}(0)}{s_0^2} = \frac{\Pi_{3B}(0)}{-s_0 c_0},\tag{8.54}$$

to be satisfied, leaving 6 of the 8 initial parameters. We now to take into account how these parameters affect the inputs. Inspecting Eq. (8.48) we see that, among the $\Pi'(0)$'s, only $\Pi_{AA}(0)$ enters and is thus determined in terms of the input parameters. Among the $\Pi(0)$'s, on the other hand, we see that $\Pi_{WW}(0)$ and $\Pi_{ZZ}(0)$ are involved. These three constraints diminish the number of independent form factors from 6 to 3. Three convenient combinations are

$$\begin{aligned}T &= \frac{\Pi_{33}(0) - \Pi_{WW}(0)}{m_W^2}, \\ S &= \frac{g}{g'} \Pi'_{3B}(0), \\ U &= \Pi'_{33}(0) - \Pi'_{WW}(0).\end{aligned}\tag{8.55}$$

The interpretation is as follows: T and U measure the amount of custodial symmetry breaking (i.e. the difference between the W^\pm and W^3 vectors) at order q^0 and q^2 , respectively. The S -parameter, on the other hand, measures the total contribution to the $W^3 - B$ kinetic mixing, and is thus a way of ‘‘counting’’ the additional contribution from new physics. Once physical observables are written in terms of the EW parameters, they can be used to put bounds on the new physics contributions.

We conclude this section commenting on what happens when the hierarchy between the SM and new physics states is not sufficiently large to justify the approximation $\Pi'(q^2) \simeq \Pi'(0)$. When this happens, it is customary to extend the expansion in Eq. (8.31) up to order $\mathcal{O}(q^4)$:

$$\Pi_{XY}(q^2) \simeq \Pi_{XY}(0) + q^2 \Pi'_{XY}(0) + \frac{(q^2)^2}{2} \Pi''_{XY}(0).\tag{8.56}$$

⁵We will get back to this point using EFTs techniques in Sec. 9.2.

This introduces 4 new form factors. The convenient definitions are

$$\begin{aligned} W &= \frac{m_W^2}{2} \Pi''_{33}(0), & Y &= \frac{m_W^2}{2} \Pi''_{BB}(0), \\ V &= \frac{m_W^2}{2} (\Pi''_{33}(0) - \Pi''_{WW}(0)), & X &= \frac{m_W^2}{2} \Pi''_{3B}(0). \end{aligned} \quad (8.57)$$

These additional form factors can also be used in the expressions for the physical observables, and bounds can be imposed from data. We will show in Sec. 9.2 that only the T , S , W and Y parameters are generated by $d = 6$ operators, while the others are generated at higher orders.

8.5 Further consequences

8.5.1 Number of active neutrino species

One of the main results of the LEP experiment is the precise measurement of the total and partial Z decay width. The total decay width is

$$\Gamma_Z = \Gamma_{qq} + \Gamma_{\ell\ell} + \Gamma_{\nu\nu}, \quad (8.58)$$

where Γ_{qq} is the sum over all hadronic (quark) channels, $\Gamma_{\ell\ell}$ is the sum over all charged leptonic channels and $\Gamma_{\nu\nu}$ is the sum over all neutrino contributions. Experimentally only charged particles can be detected, i.e. Γ_{qq} and Γ_{ee} can be directly measured. What is left is called the *invisible* decay width. There is still the problem of how to measure Γ_Z and $\Gamma_{\nu\nu}$ in an independent way.

We can use once more the optical theorem to compute the total decay width Γ_Z , since it appears in the complete Z propagator. A measurement of $\sigma(e^+e^- \rightarrow \text{hadrons})$ with $\sqrt{s} = m_Z$ (the pole Z mass) was performed at LEP I, and can be written as

$$\sigma \sim \left| \frac{1}{q^2 - m_Z^2 + im_Z \Gamma_Z} \right|_{q^2 \simeq m_Z^2} \sim \frac{1}{(q^2 - m_Z^2)^2 + m_Z^2 \Gamma_Z^2}. \quad (8.59)$$


This is the so-called Breit-Wigner distribution. The maximum corresponds to $p^2 = m_Z^2$ and can be used to extract informations about the mass of the decaying particle, while the width of the curves at half height can be used to extract Γ_Z . The current measurements are

$$\begin{aligned} \Gamma_Z &= (2.4952 \pm 0.0023) \text{ GeV}, \\ \Gamma_{ee} &= (83.984 \pm 0.086) \text{ MeV}, \\ \Gamma_{qq} &= (1744 \pm 2.0) \text{ MeV}, \\ \Gamma_{\nu\nu} &= (499.0 \pm 1.5) \text{ MeV}. \end{aligned} \quad (8.60)$$

This must be compared with the SM predictions (for one generation)

$$\begin{aligned}
 \Gamma_{ee} &= \frac{G_F m_Z^3}{3\sqrt{2}\pi} \left[\left(-\frac{1}{2} + s_w^2 \right)^2 + s_w^4 \right] \simeq 83 \text{ MeV}, \\
 \Gamma_{uu} &= \frac{3G_F m_Z^3}{3\sqrt{2}\pi} \left[\left(\frac{1}{2} - \frac{2}{3}s_w^2 \right)^2 + \frac{4}{9}s_w^4 \right] \simeq 280 \text{ MeV}, \\
 \Gamma_{dd} &= \frac{3G_F m_Z^3}{3\sqrt{2}\pi} \left[\left(-\frac{1}{2} + \frac{1}{3}s_w^2 \right)^2 + \frac{1}{9}s_w^4 \right] \simeq 370 \text{ MeV}, \\
 \Gamma_{\nu\nu} &= \frac{G_F m_Z^3}{4 \times 3\sqrt{2}\pi} \simeq 165 \text{ MeV}.
 \end{aligned} \tag{8.61}$$

For simplicity we consider only the tree-level predictions, which are sufficient for our purposes.

 **Exercise 8.6** Compute and check the formulas above.

The number of neutrinos can be extracted from

$$N_\nu = \frac{\Gamma_Z - (3\Gamma_{ee} + 3\Gamma_{dd} + 2\Gamma_{uu})}{\Gamma_{\nu\nu}} \sim 3. \tag{8.62}$$

This gives two fundamental informations: (i) there are only three *active* neutrinos (neutrinos coupled to the Z boson), and (ii) the invisible decay width of the Z boson is saturated by the neutrino contribution, leaving only a small contribution available for other invisible states that may be present in theories that extend the SM.

8.5.2 The ρ parameter

Using the loop formalism described above we can now compute the radiative corrections to the ρ parameter. The easiest way is to consider the effective loop Lagrangian in Eq. (8.40) and compute the Fermi Lagrangian along the lines of Sec. 7.7.4. Some straightforward algebra allows to deduce that

$$\Delta\rho = \frac{\text{Re} [\Pi_{WW}(0)]}{c_0^2 m_{Z_0}^2} - \frac{\text{Re} [\Pi_{ZZ}(0)]}{m_{Z_0}^2} = \frac{\text{Re} [\Pi_{WW}(0)]}{m_W^2} - \frac{\text{Re} [\Pi_{ZZ}(0)]}{m_Z^2} \tag{8.63}$$

to the lowest order in the radiative corrections. Notice that $\Delta\rho = -T$.

8.5.3 Flavor Changing Neutral Currents

Let us consider Flavor Changing Neutral Processes, i.e. processes in which the flavor of the particles is changed but not their electric charge. For historical reasons these processes are called Flavor Changing Neutral Currents (FCNC). At tree level, the only processes that can change flavor are mediated by the W bosons and involve the

CKM matrix. These are thus Flavor Changing Charged Currents, and not FCNC. An example is given by the β -decay $d \rightarrow ue\bar{\nu}_e$ considered in Sec. 7.4, in which there is a changing in the quark flavor, but also in the electric charge. At loop level the situation is different, and we can have FCNCs. An example is given by the process $b \rightarrow s\gamma$, for which the Feynman diagram is



$$(8.64)$$

where the photon line can be attached to both the W or fermion line. The only gauge invariant operator that can describe this amplitude is

$$\mathcal{O}_{b \rightarrow s\gamma} = \bar{s}_L \sigma^{\mu\nu} b_R F_{\mu\nu} \quad (8.65)$$

together with its hermitian conjugate. Interestingly, symmetry arguments allow us to infer the form of the Wilson coefficient:

- It must be proportional to the electric charge e , given the presence of the photon (remember that, with the exception of the kinetic term, the photon field always appears in the combination eA_μ , and the interactions are thus invariant under the rescaling $e \rightarrow \lambda e$, $A_\mu \rightarrow A_\mu/\lambda$);
- It must be suppressed by a loop factor. We explicitly write this as $1/(16\pi^2)$;
- It must be proportional to g^2 , since it involves two vertices with the W boson;
- Finally, it must be proportional to the bottom mass m_b . This can be understood as follows: the gauge invariant operator involves the LH s -quark and the RH b -quark. Weak interactions, on the other hand, involve s_L and b_L . To flip the chirality from b_L to b_R we need an insertion of the only operator that involves both chiralities, i.e. the b -quark mass. The same reasoning can be applied to the operator involving b_L and s_R . The coefficient in this case is proportional to $m_s \ll m_b$ and we can neglect it in first approximation.

Putting everything together we obtain

$$c_{b \rightarrow s\gamma} = \frac{g^2}{16\pi^2} \frac{e m_b}{m_W^2} \sum_i V_{ib} V_{is}^* F \left(\frac{m_i^2}{m_W^2} \right), \quad (8.66)$$

where F is the explicit loop function (called Inami-Lim function). We expect it to be $\mathcal{O}(1)$, since we have already explicitly factorized the loop factor. We have explicitly inserted a factor m_W^2 in the denominator to give the correct dimensions to the Wilson coefficient. We can make further progress observing that $m_{u,c} \ll m_W$, in such a way that for the light up-type quarks we can Taylor expand the F function, obtaining

$$F \left(\frac{m_{u,c}^2}{m_W^2} \right) \simeq F(0) + \frac{m_{u,c}^2}{m_W^2} F'(0) + \dots \quad (8.67)$$

Notice that we can always redefine $F(0) = 0$ simply shifting the F function according to $F(x) \rightarrow F(x) - F(0)$. This shift cannot affect the physics, since the $F(0)$ factor is multiplied by $\sum_i V_{ib}V_{is}^* = 0$. This result follows from unitarity. Since, on the other hand, $m_t \simeq 2m_W$, it is clear that we cannot expand in the top-quark mass, and the top contribution will be $\mathcal{O}(1)$. Let us now explicitly expand the sum in Eq. (8.66):

$$\begin{aligned} \sum_i V_{ib}V_{is}^* F\left(\frac{m_i^2}{m_W^2}\right) &\simeq V_{ub}V_{us}^* \frac{m_u^2}{m_W^2} F'(0) + V_{cb}V_{cs}^* \frac{m_c^2}{m_W^2} F'(0) + V_{tb}V_{ts}^* F\left(\frac{m_t^2}{m_W^2}\right) \\ &\sim \lambda^4 \frac{m_u^2}{m_W^2} F'(0) + \lambda^2 \frac{m_c^2}{m_W^2} F'(0) + \lambda^2 F\left(\frac{m_t^2}{m_W^2}\right) \\ &\sim \lambda^2 F\left(\frac{m_t^2}{m_W^2}\right). \end{aligned} \tag{8.68}$$

Remembering that the Cabibbo angle is small, we obtain that the dominant contribution from the top quark is very suppressed, since we expect

$$c_{b \rightarrow s \gamma} \sim \frac{g^2}{16\pi^2} \frac{e m_b}{m_W^2} \lambda^2 \tag{8.69}$$

apart from $\mathcal{O}(1)$ numbers. This suppression is often called GIM suppression, from a seminal paper by Glashow-Iliopoulos-Maiani (Phys.Rev.D 2 (1970) 1285-1292) that first observed that FCNC were suppressed in the SM by factors much more severe than a simple loop suppression. Historically, the GIM paper was published when only the u , d and s quarks were known and discussed 1-loop kaons mixings. At that time, it allowed for the prediction of the existence of a fourth quark (what would be later known as charm) to guarantee the suppression of the kaon mixing. What we have presented in this section is the modern version of the GIM mechanism, which takes into account also the top quark.

8.6 The Higgs boson

We conclude this chapter with an overview of the properties of the Higgs boson (discovered only in 2012 at the CERN-LHC by the ATLAS and CMS experiments). As we have seen in Ch. 7, the Higgs particle interactions with the other SM states are

$$\begin{aligned} \mathcal{L}_{int} &= \left(m_W^2 W^+ W^- + \frac{m_Z^2}{2} Z^2 \right) \left(1 + \frac{h}{\sqrt{2}v} \right)^2 \\ &\quad - (\bar{e}_L M_e e_R + \bar{u}_L M_u u_R + \bar{d}_L M_d d_R) \left(1 + \frac{h}{\sqrt{2}v} \right), \end{aligned} \tag{8.70}$$

i.e. they are all proportional to the mass of the state. This prediction can be tested experimentally. We show the most recent experimental results in Fig. 8.1. From the practical point of view, in order to produce or detect the Higgs boson decays at colliders,

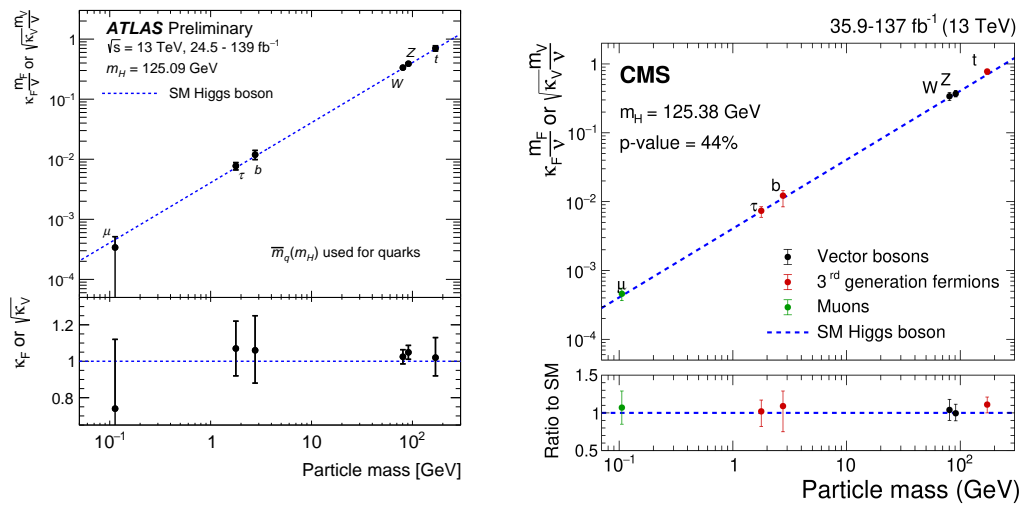


Figure 8.1: Higgs couplings versus particle mass measured in the ATLAS (left panel) and CMS (right panel) experiments at the CERN-LHC. Taken from [this](#) and [this](#) links.

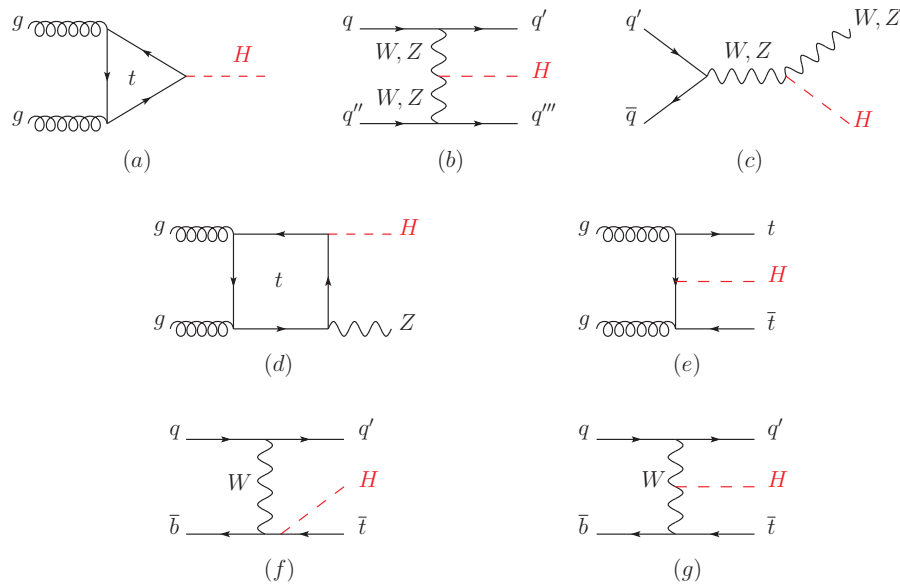


Figure 8.2: Feynman diagrams for the main Higgs boson production modes at a hadron collider: (a) gluon fusion, (b) Vector-boson fusion, (c) Higgs-strahlung (or associated production with a gauge boson at tree level from a quark-quark interaction), (d) associated production with a gauge boson (at loop level from a gluon-gluon interaction), (e) associated production with a pair of top quarks (there is a similar diagram for the associated production with a pair of bottom quarks), (f-g) production in association with a single top quark. Taken from [this](#) link.

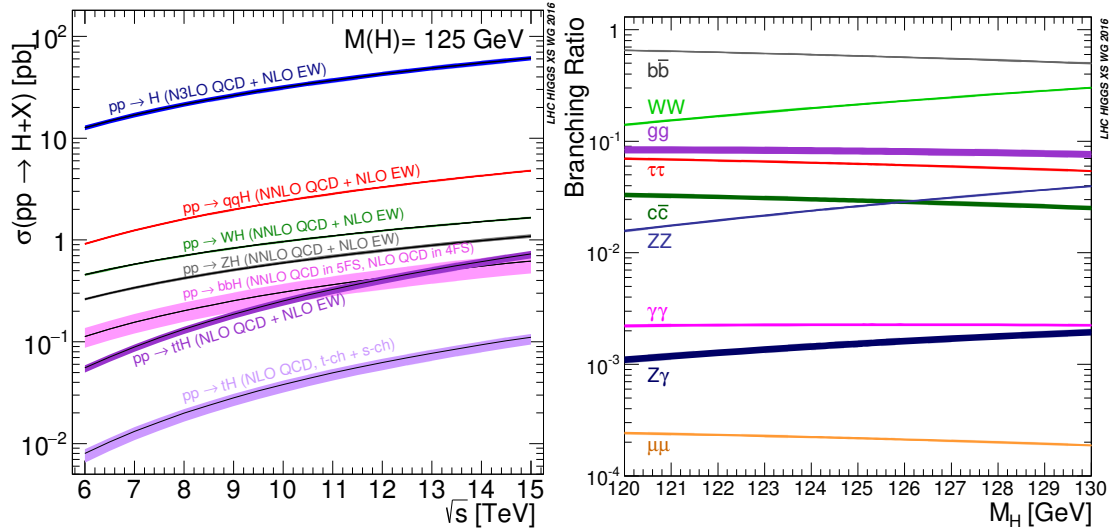


Figure 8.3: Left panel: Higgs production cross section in pp collisions for various center-of-mass energies. Right panel: Higgs branching ratios as a function of the mass. Taken from [this link](#).

the favorite channels are those involving the heaviest particles: either a $b\bar{b}$ pair or a WW^* or ZZ^* pair, in which $*$ means that one of the states is off-shell.⁶ The problem with such channels is practical: they have huge backgrounds, and are thus difficult to detect experimentally. The processes that actually allowed for the Higgs discovery in 2012 actually appear at loop level: $h \rightarrow \gamma\gamma$ and $GG \rightarrow h$. The main Higgs production mechanisms are shown in Fig. 8.2, while we present in Fig. 8.3 the total production cross section in pp collisions (left panel) and the branching ratio in the different channels (right panel). As in illustration of the current status of the measurements, we show in Fig. 8.4 the comparison between data and the SM prediction (the vertical line).

⁶This is due to the fact that $m_W = 80.4$ GeV, $m_Z = 91$ GeV while $m_h = 125$ GeV, i.e. the WW and ZZ pairs cannot contain two on-shell states by Lorentz invariance.

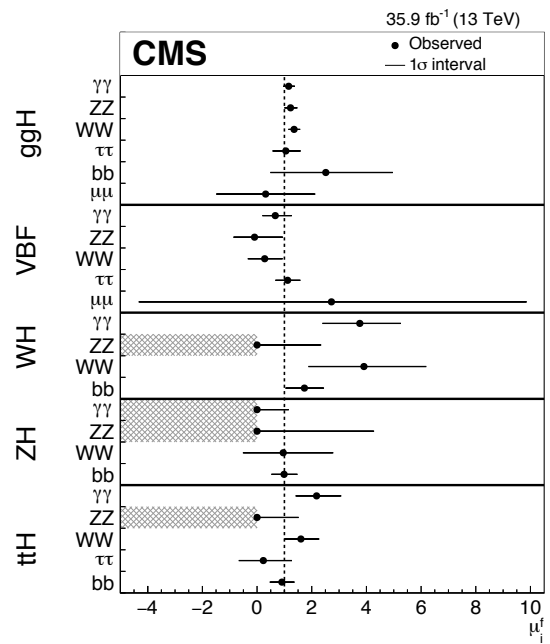
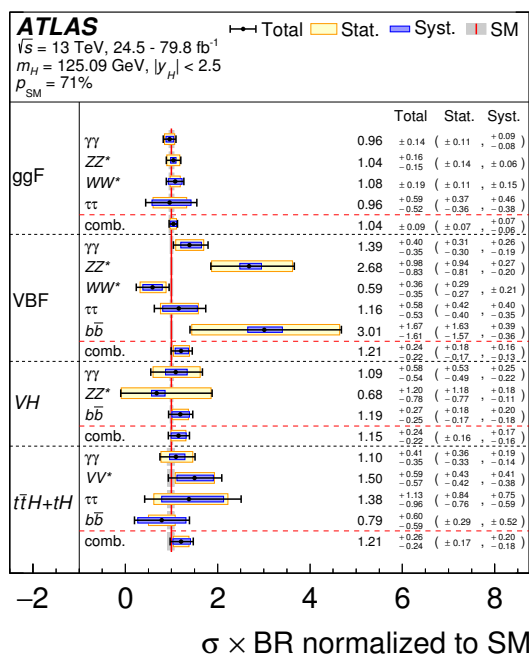


Figure 8.4: Combined measurements of the products $\sigma \cdot \text{BR}$, normalized to the SM predictions, for the five main production and five main decay modes. The hatched combinations require more data for a meaningful confidence interval to be provided. Taken from [this link](#).

Part IV

Beyond the renormalizable Standard Model

Chapter 9 Higher dimensional operators


As we already mentioned in Section 2.8, the modern view on QFT is that the renormalizable SM is only the low energy limit of a more fundamental theory, and must come with a tower of operators of increasing dimensions. Let us discuss some of the physical consequences of the higher dimensional operators.

9.1 Dimension 5

At the level of dimension 5 the only operator that can be constructed out of the particle content of the SM is

$$\mathcal{L}_5 = \frac{C_{ij}}{\Lambda} (L_i H)(L_j H), \quad (9.1)$$

where we used the two-component notation for the fermions (see App. C) and i and j are flavor indices. The operator above is called *Weinberg operator*. It is worth stressing that for our assumption to be correct we must require $\Lambda \gg v$.

 **Exercise 9.1** Show that the Weinberg operator is the only dimension 5 operator that can be constructed with the SM field content and is invariant under $SU(3)_c \times SU(2)_L \times U(1)_Y$ transformations.

Once we consider the Higgs doublet vev we obtain

$$\mathcal{L}_5 = \frac{C_{ij} v^2}{\Lambda} \nu_i \nu_j, \quad (9.2)$$

a Majorana mass term for neutrinos. The first effect we encounter in our journey through the higher dimensional operators is the generation of a mass for the only fermion that is massless in the SM. One possible interpretation of this fact is that the origin of neutrino masses is different from the origin of the other fermion masses in the SM. Moreover, we expect neutrino masses to be smaller than the masses of the other fermions, since

$$m_f \sim yv, \quad m_\nu \sim C \frac{v}{\Lambda} v \ll Cv. \quad (9.3)$$

If we look at data, we see that the mass of the lightest charged fermion (the electron) is $m_e = 0.5$ MeV, while for neutrinos we have $m_\nu \lesssim 1$ eV, i.e. there is difference of about 5 orders of magnitude between neutrino masses and the lightest charged lepton. We thus conclude that there are indications for a qualitative agreement between the expectations of the EFT and data, although it is worth stressing that there are well motivated extensions

of the SM in which neutrino masses are generated by additional physics with a mass scale below or at the EW scale. We will get back in the next chapter on possible origins of the Weinberg operator.

For the moment we just observe that the fact that neutrinos are now massive implies that we have to worry about the diagonalization of the neutrino mass matrix. The situation is now similar to the one we already encountered in Sec. 7.5.4: we have to diagonalize simultaneously two matrices in the lepton sector (the lepton Yukawa Y_e and the neutrino mass matrix $M_\nu = Cv^2/\Lambda$), with only three fields available to absorb the matrices needed to diagonalize the matrices (e_L , e_R and ν_L). The mismatch between the diagonalization matrices of Y_e and M_ν causes the appearance of the analog of the CKM matrix in the lepton sector, the so-called PMNS (Pontecorvo-Maki-Nakagawa-Sakata), generating the charged current flavor violating coupling

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} W_\mu^- \bar{e}_L \gamma^\mu U_{PMNS} \nu_L + h.c. \quad (9.4)$$

Exercise 9.2 If neutrino masses are generated by the Weinberg operator the neutrinos are Majorana particles. How this changes the counting of the parameters appearing in the PMNS matrix as compared to the counting of parameters appearing in the CKM matrix?

Another important consequence of the Weinberg operator is that it explicitly violates lepton number. We see here one of the most important facts regarding the accidental symmetries of the SM: such accidental symmetries of the $d = 4$ Lagrangian will be in general broken by the $d > 4$ operators. They are thus good approximate symmetries of nature only when $\Lambda \gg v$, and their small breaking can be searched experimentally as an indication of the presence of new physics beyond the SM. A good example are the flavor violating decays

$$\mu \rightarrow e\gamma, \quad \tau \rightarrow \mu\gamma, \quad \mu \rightarrow eee, \quad (9.5)$$

which have been (and are being) experimentally searched. For instance, the MEG experiment puts a very strong bound on the first process,

$$\text{BR}(\mu \rightarrow e\gamma) < 4.3 \times 10^{-13} \quad @90\%C.L. \quad (9.6)$$

What is the prediction for massive neutrinos? We can estimate such contribution with a reasoning analogous to the one we used in Sec. 8.5.3. We obtain

$$c_{\mu \rightarrow e\gamma} \sim e \frac{g^2}{16\pi^2} \frac{m_\mu}{m_W^2} \sum_{i=1,2,3} U_{ei} U_{\mu i}^* \left(\frac{m_{\nu_i}}{m_W} + \dots \right) \lesssim 10^{-19} \text{GeV}^{-1}, \quad (9.7)$$

where we explicitly used the fact that the coefficient must vanish in the $m_{\nu_i} \rightarrow 0$ limit to expand the last term. This is justified by the fact that in this limit lepton number is recovered, and the loop diagram must vanish because of symmetry. Using our estimate to compute the BR in the SM with massive neutrinos it is immediate to convince ourselves that the prediction is many orders of magnitude below the current experimental limit. As we are going to see in next section, however, dimension 6 operators can contribute to this observable.

9.2 Dimension 6

Many more operators appear at the level of dimension 6. The complete non-redundant list include 59 operators (without including flavor), and is extended to a list of 2499 independent operators once flavor indices are included (see arXiv:1008.4884). The analysis of all the physical effects of the $d = 6$ operators is beyond our scope, but it is interesting to list at least some consequences. One fact we must be aware of is that not all the $d = 6$ operators that can be written are independent: fields redefinitions can be used to eliminate operators in favor of others, and various “basis” of operators can be found in the literature (see for instance arXiv:1508.05895 for a translator between different basis).

9.2.1 Modification of the input parameters

One of the striking effects of the $d = 6$ operators is that they can modify the input parameters, changing the $d = 4$ SM predictions. More specifically, the operator

$$H^\dagger H (D_\mu H)^\dagger (D_\mu H) \quad (9.8)$$

gives an additional contribution to the Z boson mass, while the operator

$$(\bar{L}^i \gamma^\mu T_L^a L^j) (\bar{Q}^m \gamma^\mu T_L^a Q^n) \quad (9.9)$$

can modify the Fermi constant extraction for the correct choice of flavor indices. Clearly this happens when the coefficients give contributions to physical observables that are not too small. The great phenomenological success of the SM already tells us that such contributions, if present, must be smaller than the SM ones.

9.2.2 Electroweak parameters

Another interesting effect of the $d = 6$ operators is that they contribute to some of the EW parameters of Sec. 8.4. More specifically we have

$$\begin{aligned}
 |H^\dagger D_\mu H| & \quad (T) \\
 (H^\dagger T_L^a H) W_{\mu\nu}^a B_{\mu\nu} & \quad (S) \\
 (\partial_\rho B_{\mu\nu})^2 & \quad (Y) \\
 (\partial_\rho W_{\mu\nu}^a)^2 & \quad (W)
 \end{aligned} \tag{9.10}$$

All the other EW parameters are instead generated by higher dimensional operators. Experimental bounds on the EW parameters can be used to set limits on the Wilson coefficients of the operators listed.

9.2.3 FCNCs

Finally, we notice that there are new contributions to FCNC. For instance, $\mu \rightarrow e\gamma$ receives a contributions from

$$\bar{L}^i \sigma^{\mu\nu} H e_R^j B_{\mu\nu}, \quad \bar{L}^i \sigma^{\mu\nu} T_L^a H e_R^j W_{\mu\nu}^a, \tag{9.11}$$

and $b \rightarrow s\gamma$ receives contributions from

$$\bar{Q}^i \sigma^{\mu\nu} H d_R^j B_{\mu\nu}, \quad \bar{Q}^i \sigma^{\mu\nu} T_L^a H d_R^j W_{\mu\nu}^a. \tag{9.12}$$

Many more contributions can be listed that would affect all the sectors of the SM. Their analysis goes however beyond the scope of these lectures

Part V

Drawbacks of the Standard Model

As we saw in the previous chapters, the SM is an extremely successful theory. Nevertheless, there are some experimentally observed phenomena that are still not explained by the SM, as well as some theoretical aspects that are not clear. We devote this chapter to a brief discussion of each of the following topics:

- Experimental drawbacks:
 - Neutrino masses
 - Cosmological observations: dark matter and the matter/antimatter asymmetry
- Theoretical drawbacks:
 - The hierarchy problem
 - Charge quantization

Chapter 10 Neutrino masses

Chapter 11 Cosmological observations

11.1 Dark Matter

11.2 The matter/antimatter asymmetry

Chapter 12 The Hierarchy problem

Chapter 13 Charge quantization

Appendix Lorentz invariant Lagrangians for free particles (INCOMPLETE)

In this Appendix we will use Lorentz covariance to write the most general Lagrangians for spin 0, 1/2 and 1 particles.

A.1 Spin 0

A spin-0 particle can be embedded in a scalar field ϕ obeying

$$U^\dagger(\Lambda)\phi U(\Lambda) = \phi. \quad (\text{A.1})$$

To write down the most general Lagrangian we will limit ourselves to the lower dimensional terms. Up to quadratic order in the ϕ 's and in the derivatives we have

$$\mathcal{L} = a(\partial\phi)^2 + b(\square\phi)\phi + c\phi^2. \quad (\text{A.2})$$

Remembering that integrations by parts are allowed in the action, and that total derivative can be disregarded because they do not modify the equations of motions, we see that for the second term we have

$$\begin{aligned} b \int d^4x (\square\phi)\phi &= b \int d^4x (\partial_\mu\partial^\mu\phi)\phi \\ &= (\text{by parts}) \\ &= -b \int d^4x (\partial\phi)^2, \end{aligned} \quad (\text{A.3})$$

i.e. it is completely equivalent to the first one. We will thus keep only the a term. The equations of motions are

$$2a\square\phi - 2c\phi = 0 \quad \Rightarrow \quad \square\phi - \frac{c}{a}\phi = 0. \quad (\text{A.4})$$

This is precisely the KG equation provided we identify $c/a = -M^2$. With this information the Lagrangian results

$$\mathcal{L} = a [(\partial\phi)^2 - M^2\phi^2]. \quad (\text{A.5})$$

The constant a is not determined, although a simple computation shows that we need $a > 0$ to guarantee positive energy. It is conventional to fix $a = 1/2$, and we obtain

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{M^2}{2}\phi^2. \quad (\text{A.6})$$

A.2 Spin 1/2

The theory of Lorentz group tells us that the transformations of left handed (LH) and right handed (RH) fermions are

$$\begin{aligned}\psi_L &\rightarrow e^{-i(\boldsymbol{\alpha}+i\boldsymbol{\beta})\boldsymbol{\sigma}/2}\psi_L \equiv U_L\psi_L, \\ \psi_R &\rightarrow e^{-i(\boldsymbol{\alpha}-i\boldsymbol{\beta})\boldsymbol{\sigma}/2}\psi_R \equiv U_R\psi_R.\end{aligned}\tag{A.7}$$

The parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ parametrize the boost and rotations. In terms of the usual transformation rules of a 4-vector $V^\mu = (V^0, \mathbf{V})$ we have

$$\begin{aligned}\delta V^0 &= -\boldsymbol{\beta} \cdot \mathbf{V}, \\ \delta V^i &= -\beta^i V^0 + \epsilon^{ijk} \alpha^j V^k.\end{aligned}\tag{A.8}$$

Notice that the $U_{L,R}$ matrices are not unitary, but we have

$$U_L^{-1} = U_R^\dagger, \quad U_R^{-1} = U_L^\dagger.\tag{A.9}$$

Another important identity involves the determinant of a matrix written in terms of the Levi-Civita symbol:

$$\epsilon^{i_1 \dots i_N} U_{i_1}^{m_1} \dots U_{i_N}^{m_N} = \det U \epsilon^{m_1 \dots m_N}.\tag{A.10}$$

Specializing to the 2-dimensional representation of the Lorentz group we have that the product $\psi_L \psi_L \equiv \epsilon^{ab} \psi_L^a \psi_L^b$ is Lorentz invariant:

$$\epsilon^{ab} \psi_L^a \psi_L^b \rightarrow \epsilon^{ab} U_L^{ad} U_L^{be} \psi_L^d \psi_L^e = \epsilon^{de} \psi_L^d \psi_L^e\tag{A.11}$$

We thus conclude that the products

$$\psi_R^\dagger \psi_L, \quad \psi_L^\dagger \psi_R, \quad \psi_L \psi_L, \quad \psi_R \psi_R,\tag{A.12}$$

are all Lorentz invariant. It is now easy to show that the 4-component object

$$(\psi_L^\dagger \psi_L, \psi_L^\dagger \boldsymbol{\sigma} \psi_L)\tag{A.13}$$

transforms as a 4-vector, and can thus be combined with a derivative to form a Lorentz invariant object. To make the notation more compact we define $\bar{\sigma}^\mu = (\mathbf{1}, -\boldsymbol{\sigma})$ to rewrite

$$(\psi_L^\dagger \psi_L, \psi_L^\dagger \boldsymbol{\sigma} \psi_L) = \psi_L^\dagger \bar{\sigma}^\mu \psi_L.\tag{A.14}$$

The case of the RH spinors can be treated in a similar fashion. In this case the 4-vector is given by

$$i\psi_R^\dagger \sigma^\mu \psi_R, \quad \sigma^\mu = (\mathbf{1}, \boldsymbol{\sigma}).\tag{A.15}$$

It is interesting to observe that any RH fermion can be converted to a LH fermion. The key observation is

$$\epsilon \sigma_i \epsilon = \sigma_i^* \quad \Rightarrow \quad U_R^* = -\epsilon U_L \epsilon\tag{A.16}$$

where $\epsilon \equiv i\sigma_2$. Comparing with Eq. (A.7) shows immediately that $\epsilon\psi_R^*$ transforms as a LH fermion:

$$\begin{aligned}\epsilon\psi_R^* &\rightarrow \epsilon U_R^* \psi_R^* \\ &= -\epsilon^2 U_L \epsilon \psi_R^* \\ &= U_L \epsilon \psi_R^*.\end{aligned}\tag{A.17}$$

We can thus just work with one type of spinor. The notation commonly employed is the one of Weyl fermions:

$$\psi \equiv \psi_L, \quad \psi^c \equiv \epsilon\psi_R^*.\tag{A.18}$$

The last expression can be inverted as $\psi_R = -\epsilon(\psi^c)^* \equiv -\epsilon\bar{\psi}^c$, implying that the invariant product $\psi_R^\dagger\psi_L$ can be written as

$$\psi_R^\dagger\psi_L = \psi^c\epsilon\psi \equiv \psi^c\psi = \psi\psi^c.\tag{A.19}$$

In the last step we have used the antisymmetry of ϵ combined with the anticommuting nature of the spinor fields. The other combination can be written as

$$\psi_L^\dagger\psi_R = -\bar{\psi}\epsilon\bar{\psi}^c \equiv \bar{\psi}\bar{\psi}^c = \bar{\psi}^c\bar{\psi}.\tag{A.20}$$

Notice that the definition of the product with the ‘‘barred’’ quantities differs by a minus sign from the definition of the product with ‘‘unbarred’’ quantities. As for the 4-vectors, an explicit computation shows that

$$\psi_R^\dagger\sigma^\mu\psi_R = (\psi^{c\dagger}\bar{\sigma}^\mu\psi^c)^*.\tag{A.21}$$

This completely solves the problem of the Lorentz invariants. We can write the generic Lagrangian as

$$\mathcal{L} = i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + i\psi^{c\dagger}\bar{\sigma}^\mu\partial_\mu\psi^c + c\psi\psi + d\psi^c\psi^d + e\psi^c\psi + h.c.\tag{A.22}$$

for generic complex coefficients c , d , and e . The equations of motion can be computed applying $\delta/\delta\psi^\dagger$ and $\delta/\delta\psi^{c\dagger}$. These derivatives are themselves anticommuting, a fact that must be taken into account when deriving products. We obtain

$$\begin{aligned}i\bar{\sigma}^\mu\partial_\mu\psi + 2c^*\epsilon\psi^* + e^*\epsilon\psi^{c*} &= 0, \\ i\bar{\sigma}^\mu\partial_\mu\psi^c + 2d^*\epsilon\psi^{c*} + e^*\epsilon\psi^* &= 0.\end{aligned}\tag{A.23}$$

Observing now that $(\sigma \cdot \partial)(\bar{\sigma} \cdot \partial) = \square$ and applying the proper operator to the equations of motion we obtain

$$\begin{aligned}\square\psi + 4|c|^2\psi^c + 2(c^*e + e^*c)\psi^c + |e|^2\psi &= 0, \\ \square\psi^c + 4|d|^2\psi^c + 2(d^*e + e^*d)\psi + |e|^2\psi^c &= 0.\end{aligned}\tag{A.24}$$

To interpret these equations we first consider what happens if only one between ψ or ψ^c is present in the theory. When only ψ is present we obtain

$$(\square + 4|c|^2)\psi = 0,\tag{A.25}$$

which is a KG equation. We thus interpret c as a mass parameter. More precisely, we identify

$$c \equiv -\frac{m_\psi}{2}, \quad (\text{A.26})$$

where m is called the Majorana mass of the ψ particle. A reasoning completely analogous applies when only ψ^c is present, leading to identity

$$d \equiv -\frac{m_{\psi^c}}{2} \quad (\text{A.27})$$

with the Majorana mass for the ψ^c particle. Notice that Majorana mass terms in the Lagrangian can be forbidden by symmetries (for instance by a $U(1)$ symmetry). Let us now suppose this is the case and that Majorana masses are not present in \mathcal{L} . If only one between ψ or ψ^c is present (i.e. $e = 0$) then the state is necessarily massless,

$$\square\psi = 0 \quad \text{or} \quad \square\psi^c = 0. \quad (\text{A.28})$$

If on the other hand the $\psi^c\psi$ term is present, then we obtain

$$(\square + |e|^2)\psi = 0, \quad \text{and} \quad (\square + |e|^2)\psi^c = 0. \quad (\text{A.29})$$

In this case we identify e with another type of mass, called Dirac mass:

$$e = -M. \quad (\text{A.30})$$

Notice that ψ and ψ^c are degenerate in mass. This is the familiar case of a Dirac fermion (called in this way precisely because it has a non-vanishing Dirac mass). To make contact with the usual notation, we define a set of 4 gamma matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad (\text{A.31})$$

and the Dirac 4-component spinor as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \bar{\psi} = \psi^\dagger \gamma^0, \quad (\text{A.32})$$

to write

$$\begin{aligned} \mathcal{L}_{Dirac} &= i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi + i\psi^{c\dagger} \bar{\sigma}^\mu \partial_\mu \psi^c - M\psi^c\psi + h.c. \\ &= \bar{\psi} + i\gamma^\mu \partial_\mu \psi - M\bar{\psi}\psi. \end{aligned} \quad (\text{A.33})$$

It is a simple computation to show the equivalence between the two expressions.

Finally, we discuss the case in which both Majorana and Dirac masses are present. To clarify the meaning of the equations of motion in this case it is convenient to define $(\psi, \psi^c)^T$ and write

$$\begin{pmatrix} \square & 0 \\ 0 & \square \end{pmatrix} \begin{pmatrix} \psi \\ \psi^c \end{pmatrix} + \begin{pmatrix} |M|^2 + |m_\psi|^2 & m_\psi^* M + m_\psi M^* \\ m_{\psi^c}^* M + m_{\psi^c} M^* & |M|^2 + |m_{\psi^c}|^2 \end{pmatrix} \begin{pmatrix} \psi \\ \psi^c \end{pmatrix} = 0. \quad (\text{A.34})$$

The matrix containing the mass terms can be written as $\mathcal{M}^\dagger \mathcal{M}$, where

$$\mathcal{M} = \begin{pmatrix} m_\psi & M \\ M & m_{\psi^c} \end{pmatrix}. \quad (\text{A.35})$$

The source of the confusion in the interpretation of the mass matrix is due to the fact that the mass matrix \mathcal{M} is not diagonal. The idea to give a physical interpretation in this case is the following: neither the ψ nor the ψ^c field represents a physical state with well-defined mass. To obtain fields representing physical particles we redefine $(\psi, \psi^c)^T$ in such a way that the mass term becomes diagonal. This can be achieved in the following way: since the $\mathcal{M}^\dagger \mathcal{M}$ matrix is hermitian, it can be diagonalized by a unitary transformation:

$$U^\dagger \mathcal{M}^\dagger \mathcal{M} U = \mathcal{M}_{diag}^2 = \begin{pmatrix} M_1^2 & 0 \\ 0 & M_2^2 \end{pmatrix}, \quad (\text{A.36})$$

where $M_{1,2}$ are real and positive parameters. Defining now

$$\begin{pmatrix} \psi \\ \psi^c \end{pmatrix} = U \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (\text{A.37})$$

we obtain the equation of motion

$$\begin{pmatrix} \square & 0 \\ 0 & \square \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \begin{pmatrix} M_1^2 & 0 \\ 0 & M_2^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \quad (\text{A.38})$$

The fields $\psi_{1,2}$ now represent particles of well defined mass, $M_{1,2}$, respectively. Can all this can be inferred directly from the Lagrangian? Focussing on the mass term only we have

$$\begin{aligned} -\mathcal{L}_{mass} &= \frac{m_\psi}{2} \psi^2 + \frac{m_{\psi^c}}{2} (\psi^c)^2 + M \psi^c \psi + h.c. \\ &= \frac{1}{2} (\psi, \psi^c)^T \mathcal{M} \begin{pmatrix} \psi \\ \psi^c \end{pmatrix}. \end{aligned} \quad (\text{A.39})$$

The matrix \mathcal{M} can thus be read off directly from the Lagrangian. Notice however that there is an obstacle to the diagonalization of such matrix: \mathcal{M} is a complex (non-hermitian) matrix, so it is not obvious whether it can be diagonalized by some unitary transformation (recall that we need unitary transformations not to mess up with the canonical form of the kinetic terms). To understand this point, we go back to Eq. (A.36) and observe that there is a second hermitian matrix that can be constructed out of \mathcal{M} , namely $\mathcal{M} \mathcal{M}^\dagger$. In general

$$[\mathcal{M}^\dagger \mathcal{M}, \mathcal{M} \mathcal{M}^\dagger] \neq 0, \quad (\text{A.40})$$

implying that $\mathcal{M} \mathcal{M}^\dagger$ can be diagonalized by a unitary matrix which is different from U .

We will call such matrix V :

$$V^\dagger \mathcal{M} \mathcal{M}^\dagger V = \mathcal{M}_{diag}^2 = \begin{pmatrix} M_1^2 & 0 \\ 0 & M_2^2 \end{pmatrix}. \quad (\text{A.41})$$

The eigenvalues of $\mathcal{M} \mathcal{M}^\dagger$ are the same as those of $\mathcal{M}^\dagger \mathcal{M}$, since $\det(\mathcal{M} \mathcal{M}^\dagger - \lambda) = \det(\mathcal{M}^\dagger \mathcal{M} - \lambda)$. Combining the diagonalization equations for $\mathcal{M}^\dagger \mathcal{M}$ and $\mathcal{M} \mathcal{M}^\dagger$ we obtain

$$V^\dagger \mathcal{M} U = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \equiv \mathcal{M}_{diag}. \quad (\text{A.42})$$

We conclude that *any complex matrix can be diagonalized by a bi-unitary transformation*. In the fermion case under consideration we can say something more, since the \mathcal{M} matrix is symmetric:

$$\mathcal{M} = \mathcal{M}^T \quad \Rightarrow \quad \mathcal{M}^* = \mathcal{M}^\dagger. \quad (\text{A.43})$$

Using the fact that $\mathcal{M}_{diag}^\dagger = \mathcal{M}_{diag} = \mathcal{M}_{diag}^*$ we obtain

$$\begin{aligned} \mathcal{M}_{diag} &= \mathcal{M}_{diag}^\dagger = U^\dagger \mathcal{M}^\dagger V = U^\dagger \mathcal{M}^* V \\ &= \mathcal{M}_{diag}^* = V^T \mathcal{M}^* U^* \end{aligned} \quad (\text{A.44})$$

The two expressions are compatible only if $V^T = U^\dagger$. This concludes our derivation: any symmetric matrix \mathcal{M} can be diagonalized by a unitary transformation

$$U^T \mathcal{M} U = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \equiv \mathcal{M}_{diag}. \quad (\text{A.45})$$

Going back to the mass Lagrangian and redefining the fields according to Eq. (A.37) we obtain

$$\begin{aligned} -\mathcal{L}_{mass} &= \frac{m_\psi}{2} \psi^2 + \frac{m_{\psi^c}}{2} (\psi^c)^2 + M \psi^c \psi + h.c. \\ &= \frac{1}{2} (\psi, \psi^c)^T \mathcal{M} \begin{pmatrix} \psi \\ \psi^c \end{pmatrix} \\ &= \frac{1}{2} (\psi_1, \psi_2)^T \mathcal{M}_{diag} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \end{aligned} \quad (\text{A.46})$$

which is now written completely in terms of mass eigenstates. Needless to say, since the transformation we are performing on the states is unitary, the canonical form of the kinetic term is exactly preserved.

A.3 Spin 1

The Lorentz transformation of a massive spin-1 field is

$$U^\dagger V^\alpha U = \Lambda^\alpha_\beta V^\beta. \quad (\text{A.47})$$

To construct a Lorentz invariant Lagrangian is thus sufficient to correctly contract the indices. The most general Lagrangian containing at most two fields and two derivatives is

$$\mathcal{L} = aV_\mu \square V^\mu + bV_\mu \partial^\mu \partial^\alpha V_\alpha + cV_\mu V^\mu. \quad (\text{A.48})$$

A possible term $(\partial_\mu V^\mu)^2$ has been integrated by parts and contributes to the second term. The equation of motion in position space is

$$[(a\square + c)g^{\mu\alpha} + b\partial^\mu \partial^\alpha] V_\alpha = 0. \quad (\text{A.49})$$

This equation can be converted to momentum space using

$$V_\alpha = \epsilon_\alpha e^{-ikx}. \quad (\text{A.50})$$

We obtain

$$(ak^2 - c)\epsilon_\alpha + b(k \cdot \epsilon)k_\alpha = 0. \quad (\text{A.51})$$

This equation has two kinds of solutions:

- **TRANSVERSE:** $k \cdot \epsilon = 0$. Identifying $c/a = m_V^2$ we obtain the KG equation

$$(k^2 - m_V^2)\epsilon_\alpha^\perp = 0. \quad (\text{A.52})$$

Since V has 4 components and $k \cdot \epsilon = 0$ is one constraint, we have three possible ϵ^\perp solutions;

- **LONGITUDINAL:** $\epsilon_\alpha \propto k_\alpha$. In this case we obtain a KG equation identifying $c/(a + b) = m_L^2$.

Notice that the two solutions above correspond to the decomposition

$$V = V^\perp + \partial\alpha(x). \quad (\text{A.53})$$

What we want to do now is to eliminate the scalar degree of freedom $\alpha(x)$ from the 4-components vector field. This can be achieved setting $a + b = 0$, since for this choice there is no longitudinal solution. With this choice it is easy to determine the final form of the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}(V_{\mu\nu})^2 + \frac{m_V^2}{2}(V)^2. \quad (\text{A.54})$$

The final form is obtained choosing conventionally $a = 1/2$. As expected, the massive vector field has 3 independent components, one for each of the three independent transverse polarizations.

Appendix More details on the baryon quark-spin content (INCOMPLETE)

For a good discussion of what follows in the context of isospin see [this lectures](#).

We start by observing once more that the three quark state must be completely antisymmetric, being the quarks fermions. As we saw in Eq. (4.44), the color part is already antisymmetric. This means that the other quantum numbers must organize themselves in a symmetric way. The other quantum numbers that we must consider are spin and flavor $SU(3)_F$. Let us start from the spin. We know how to combine three spin 1/2 particles, and we immediately obtain

$$\mathbf{2} \times \mathbf{2} \times \mathbf{2} = \mathbf{2} + \mathbf{2} + \mathbf{4} . \quad (\text{B.1})$$

We obtain a spin 3/2 object (the 4 representation) and two spin 1/2 objects (the two 2 representations). We can gain further insight looking at the symmetry properties of the states using the Clebsh-Gordan coefficients:¹

	$S_z = 3/2$	1/2	-1/2	-3/2
$S = 3/2$	$ \uparrow\uparrow\uparrow\rangle$	$\frac{ \uparrow\uparrow\downarrow\rangle + \uparrow\downarrow\uparrow\rangle + \downarrow\uparrow\uparrow\rangle}{\sqrt{3}}$	$\frac{ \uparrow\downarrow\downarrow\rangle + \downarrow\uparrow\downarrow\rangle + \downarrow\downarrow\uparrow\rangle}{\sqrt{3}}$	$ \downarrow\downarrow\downarrow\rangle$
$S = 1/2$		$\psi_+^S(\mathcal{S}) = \frac{(\uparrow\uparrow\downarrow\rangle + \uparrow\downarrow\uparrow\rangle) - 2 \downarrow\uparrow\uparrow\rangle}{\sqrt{6}}$	$\psi_-^S(\mathcal{S}) = -\frac{(\downarrow\uparrow\downarrow\rangle + \downarrow\downarrow\uparrow\rangle) - 2 \uparrow\downarrow\downarrow\rangle}{\sqrt{6}}$	
$S = 1/2$		$\psi_+^S(\mathcal{A}) = \frac{ \uparrow\uparrow\downarrow\rangle - \uparrow\downarrow\uparrow\rangle}{\sqrt{2}}$	$\psi_-^S(\mathcal{A}) = \frac{ \downarrow\uparrow\downarrow\rangle - \downarrow\downarrow\uparrow\rangle}{\sqrt{2}}$	

(B.2)

We see that the spin 3/2 states are completely symmetric under exchange of any pair of quarks. The same is not true for the spin 1/2 states, for which $\psi_{\pm}^S(\mathcal{S})$ are symmetric and $\psi_{\pm}^S(\mathcal{A})$ are antisymmetric under exchange of the last two quarks. These states are known as “mixed symmetric” states.

We can already draw some conclusion. Since the spin 3/2 states are completely symmetric in spin, they must be completely symmetric in $SU(3)_F$ as well. The only completely symmetric representation in Eq. (4.30) is the 10: *spin 3/2 baryons must come in a flavor decuplet*. This is exactly what is observed in nature.

The case of the spin 1/2 baryons is more complicated, since we need to combine states that are mixed symmetric in both spin and flavor to form a completely symmetric combination. To establish our notation, we will call P_{ij} the operator that permutes the

¹For a thorough discussion, see [these lectures](#).

ij pair of indices. For the spin states we immediately have

$$P_{23}\psi_{\pm}^S(\mathcal{S}) = \psi_{\pm}^S(\mathcal{S}) , \quad P_{23}\psi_{\pm}^S(\mathcal{A}) = -\psi_{\pm}^S(\mathcal{A}) . \quad (\text{B.3})$$

Turning to the flavor states, we know that the only objects with mixed symmetry properties are the two $\mathbf{8}$ representations of Eqs. (4.28)-(4.30). We will call $\psi^F(\mathcal{S})$ the mixed symmetric combination and $\psi^F(\mathcal{A})$ the mixed antisymmetric one. They obey

$$P_{23}\psi^F(\mathcal{S}) = \psi^F(\mathcal{S}) , \quad P_{23}\psi^F(\mathcal{A}) = -\psi^F(\mathcal{A}) . \quad (\text{B.4})$$

We now apply the permutation P_{12} to start the construction of the completely antisymmetric state. For the spin states we have

$$P_{12}\psi^S(\mathcal{S}) = -\frac{1}{2}\psi^S(\mathcal{S}) + \frac{\sqrt{3}}{2}\psi^S(\mathcal{A}) , \quad P_{12}\psi^S(\mathcal{A}) = \frac{\sqrt{3}}{2}\psi^S(\mathcal{S}) - \frac{1}{2}\psi^S(\mathcal{A}) , \quad (\text{B.5})$$

while for the flavor states we obtain

$$P_{12}\psi^F(\mathcal{S}) = -2\psi^F(\mathcal{S}) + \frac{3}{2}\psi^F(\mathcal{A}) \quad (\text{B.6})$$

Appendix The Standard Model with Weyl fermions

Since the SM is a chiral theory, it would be natural to write its Lagrangian using Weyl spinors, instead of Dirac spinors. This is what we will do in this Appendix. We remind the reader of a result that we have derived in App. A: the RH fermion ψ_R can be written in terms of a LH fermion as $\psi_R = -\epsilon\bar{\psi}^c$. Let us now introduce the van der Waerden notation: we assign an index $\psi_L \rightarrow \psi_\alpha$ to the LH fields, and a different ‘‘dotted’’ index $\psi_R = -\epsilon\bar{\psi}^c \rightarrow \bar{\psi}^{c\dot{\alpha}}$ to those fields that correspond to the RH transformations. In terms of these fields we also define the invariant products

$$\psi\chi = \psi^\alpha\chi_\alpha, \quad \bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}. \quad (\text{C.1})$$

Notice that these expressions are consistent with Eqs. (A.19) - (A.20): with the new notation we obtain

$$\begin{aligned} \bar{\psi}\psi &= \begin{pmatrix} \bar{\psi}_{\dot{\alpha}} & \psi^{c\dot{\alpha}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{c\dot{\alpha}} \end{pmatrix} \\ &= \bar{\psi}_{\dot{\alpha}}\bar{\psi}^{c\dot{\alpha}} + \psi^{c\dot{\alpha}}\psi_\alpha \\ &= \bar{\psi}\bar{\psi}^c + \psi\psi^c. \end{aligned} \quad (\text{C.2})$$

In terms of the van der Waerden notation any Dirac spinor can be written as

$$\psi_L = \begin{pmatrix} \psi \\ 0 \end{pmatrix}, \quad \psi_R = \begin{pmatrix} 0 \\ \bar{\psi}^c \end{pmatrix} \quad (\text{C.3})$$

This implies that the ψ representations will match those of ψ_L , while the representations of ψ^c will be the conjugate of those of ψ_R . The representations of the SM particle content under $SU(3)_c \times SU(2)_L \times U(1)_Y$ are thus

$$\begin{aligned} L &\sim (\mathbf{1}, \mathbf{2})_{-1/2}, \\ Q &\sim (\mathbf{3}, \mathbf{2})_{1/6}, \\ e^c &\sim ((\mathbf{1}), \mathbf{1})_{+1}, \\ u^c &\sim (\bar{\mathbf{3}}, \mathbf{1})_{-2/3}, \\ d^c &\sim (\bar{\mathbf{3}}, \mathbf{1})_{1/3}. \end{aligned} \quad (\text{C.4})$$

The SM Lagrangian has precisely the same form as before,

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{fermions} + \mathcal{L}_H + \mathcal{L}_{Yuk}. \quad (\text{C.5})$$

The Higgs and gauge Lagrangians are not affected by the use of Weyl fermions, while the other two terms can now be written as

$$\begin{aligned}\mathcal{L}_{fermions} &= \mathbf{L}^\dagger i\bar{\sigma}^\mu D_\mu \mathbf{L} + \mathbf{Q}^\dagger i\bar{\sigma}^\mu D_\mu \mathbf{Q} + e^{c\dagger} i\bar{\sigma}^\mu D_\mu e^c + \mathbf{u}^{c\dagger} i\bar{\sigma}^\mu D_\mu \mathbf{u}^c + \mathbf{d}^{c\dagger} i\bar{\sigma}^\mu D_\mu \mathbf{d}^c, \\ \mathcal{L}_{Yuk} &= H^\dagger \mathbf{L} Y_e e^c + H^\dagger \mathbf{Q} Y_d \mathbf{d}^c + H^T \epsilon \mathbf{Q} Y_u \mathbf{u}^c.\end{aligned}\tag{C.6}$$

Everything can now be repeated as we did with Dirac fermions.

Appendix The optical theorem

Appendix A collection of loop integrals

We collect in this appendix some useful expressions involving 1-loop integrals. We draw heavily from the very nice review 0709.1075. Historically, the decomposition is based on the work of Passarino and Veltman (Nucl. Phys. **B160** (1979) 151), 't Hooft and Veltman (Nucl. Phys. **B153** (1979) 365), and Melrose (Nuovo Cimento **XL A** (1965) 181).

From the practical point of view it is possible to compute the loop integrals using automated tools like `Package X` or `FeynCalc`, with the loop functions listed below.

E.1 Scalar one-point function

The scalar one-point function reads

$$A_0(m) = -m^2 \left(\frac{m^2}{4\pi\mu^2} \right)^{\frac{D-4}{2}} \Gamma\left(1 - \frac{D}{2}\right) = m^2 \left(\Delta - \log \frac{m^2}{\mu^2} + 1 \right) + O(D-4), \quad (\text{E.1})$$

with the UV-divergence contained in

$$\Delta = \frac{2}{4-D} - \gamma_E + \log 4\pi \quad (\text{E.2})$$

and γ_E is Euler's constant. The terms of order $O(D-4)$ are only relevant for two- or higher-loop calculations.

E.2 Scalar two-point function

The two-point function is given by

$$\begin{aligned} B_0(p_{10}, m_0, m_1) &= \Delta - \int_0^1 dx \log \frac{[p_{10}^2 x^2 - x(p_{10}^2 - m_0^2 + m_1^2) + m_1^2 - i\varepsilon]}{\mu^2} + O(D-4) \\ &= \Delta + 2 - \log \frac{m_0 m_1}{\mu^2} + \frac{m_0^2 - m_1^2}{p_{10}^2} \log \frac{m_1}{m_0} - \frac{m_0 m_1}{p_{10}^2} \left(\frac{1}{r} - r \right) \log r \\ &\quad + O(D-4), \end{aligned} \quad (\text{E.3})$$

where r and $\frac{1}{r}$ are determined from

$$x^2 + \frac{m_0^2 + m_1^2 - p_{10}^2 - i\varepsilon}{m_0 m_1} x + 1 = (x+r) \left(x + \frac{1}{r} \right). \quad (\text{E.4})$$

The variable r never crosses the negative real axis even for complex physical masses (m^2 has a negative imaginary part!). For $r < 0$ the $i\varepsilon$ prescription yields $\text{Im } r = \varepsilon \text{sgn}(r - \frac{1}{r})$. Consequently the result (E.3) is valid for arbitrary physical parameters.

E.3 Scalar three-point function

The general result for the scalar three-point function valid for all real momenta and physical masses can be brought into the symmetric form

$$\begin{aligned}
 C_0(p_{10}, p_{20}, m_0, m_1, m_2) &= \\
 &- \int_0^1 dx \int_0^x dy [p_{21}^2 x^2 + p_{10}^2 y^2 + (p_{20}^2 - p_{10}^2 - p_{21}^2)xy \\
 &\quad + (m_1^2 - m_2^2 - p_{21}^2)x + (m_0^2 - m_1^2 + p_{21}^2 - p_{20}^2)y + m_2^2 - i\varepsilon]^{-1} \quad (\text{E.5}) \\
 &= \frac{1}{\alpha} \sum_{i=0}^2 \left\{ \sum_{\sigma=\pm} \left[\text{Li}_2\left(\frac{y_{0i} - 1}{y_{i\sigma}}\right) - \text{Li}_2\left(\frac{y_{0i}}{y_{i\sigma}}\right) \right. \right. \\
 &\quad \left. \left. + \eta\left(1 - x_{i\sigma}, \frac{1}{y_{i\sigma}}\right) \log \frac{y_{0i} - 1}{y_{i\sigma}} - \eta\left(-x_{i\sigma}, \frac{1}{y_{i\sigma}}\right) \log \frac{y_{0i}}{y_{i\sigma}} \right] \right. \\
 &\quad \left. - \left[\eta(-x_{i+}, -x_{i-}) - \eta(y_{i+}, y_{i-}) - 2\pi i \theta(-p_{jk}^2) \theta(-\Im(y_{i+} y_{i-})) \right] \log \frac{1 - y_{i0}}{-y_{i0}} \right\},
 \end{aligned}$$

with $(i, j, k = 0, 1, 2$ and cyclic)

$$\begin{aligned}
 y_{0i} &= \frac{1}{2\alpha p_{jk}^2} [p_{jk}^2(p_{jk}^2 - p_{ki}^2 - p_{ij}^2 + 2m_i^2 - m_j^2 - m_k^2) \\
 &\quad - (p_{ki}^2 - p_{ij}^2)(m_j^2 - m_k^2) + \alpha(p_{jk}^2 - m_j^2 + m_k^2)], \\
 x_{i\pm} &= \frac{1}{2p_{jk}^2} [p_{jk}^2 - m_j^2 + m_k^2 \pm \alpha_i], \\
 y_{i\pm} &= y_{0i} - x_{i\pm}, \quad (\text{E.6}) \\
 \alpha &= \kappa(p_{10}^2, p_{21}^2, p_{20}^2), \\
 \alpha_i &= \kappa(p_{jk}^2, m_j^2, m_k^2) (1 + i\varepsilon p_{jk}^2),
 \end{aligned}$$

and κ is the Källén function

$$\kappa(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 2(xy + yz + zx)}. \quad (\text{E.7})$$

The dilogarithm or Spence function $\text{Li}_2(x)$ is defined as

$$\text{Li}_2(x) = - \int_0^1 \frac{dt}{t} \log(1 - xt), \quad |\arg(1 - x)| < \pi. \quad (\text{E.8})$$

The η -function compensates for cut crossings on the Riemann-sheet of the logarithms and dilogarithms. For a, b on the first Riemann sheet it is defined by

$$\log(ab) = \log(a) + \log(b) + \eta(a, b). \quad (\text{E.9})$$

All η -functions in (E.5) vanish if α and all the masses m_i are real. Note that α is real in particular for all on-shell decay and scattering processes.

E.4 Scalar four-point function

The scalar four-point function $D_0(p_{10}, p_{20}, p_{30}, m_0, m_1, m_2, m_3)$ can be expressed in terms of 16 dilogarithms.

Before we give the result we first introduce some useful variables and functions.

We define

$$k_{ij} = \frac{m_i^2 + m_j^2 - p_{ij}^2}{m_i m_j}, \quad i, j = 0, 1, 2, 3, \quad (\text{E.10})$$

and r_{ij} and \tilde{r}_{ij} by

$$x^2 + k_{ij}x + 1 = (x + r_{ij})(x + 1/r_{ij}), \quad (\text{E.11})$$

and

$$x^2 + (k_{ij} - i\varepsilon)x + 1 = (x + \tilde{r}_{ij})(x + 1/\tilde{r}_{ij}). \quad (\text{E.12})$$

Note that for real k_{ij} the r_{ij} 's lie either on the real axis or on the complex unit circle.

Furthermore

$$P(y_0, y_1, y_2, y_3) = \sum_{0 \leq i < j \leq 3} k_{ij} y_i y_j + \sum_{j=0}^3 y_j^2, \quad (\text{E.13})$$

$$Q(y_0, y_1, 0, y_3) = (1/r_{02} - r_{02})y_0 + (k_{12} - r_{02}k_{01})y_1 + (k_{23} - r_{02}k_{03})y_3, \quad (\text{E.14})$$

$$\bar{Q}(y_0, 0, y_2, y_3) = (1/r_{13} - r_{13})y_3 + (k_{12} - r_{13}k_{23})y_2 + (k_{01} - r_{13}k_{03})y_0, \quad (\text{E.15})$$

and $x_{1,2}$ is defined by

$$\begin{aligned} & \frac{r_{02}r_{13}}{x} \left\{ \left[P\left(1, \frac{x}{r_{13}}, 0, 0\right) - i\varepsilon \right] \left[P\left(0, 0, \frac{1}{r_{02}}, x\right) - i\varepsilon \right] \right. \\ & \quad \left. - \left[P\left(0, \frac{x}{r_{13}}, \frac{1}{r_{02}}, 0\right) - i\varepsilon \right] \left[P\left(1, 0, 0, x\right) - i\varepsilon \right] \right\} \\ & = ax^2 + bx + c + i\varepsilon d = a(x - x_1)(x - x_2), \end{aligned} \quad (\text{E.16})$$

where

$$\begin{aligned} a &= k_{23}/r_{13} + r_{02}k_{01} - k_{03}r_{02}/r_{13} - k_{12}, \\ b &= (r_{13} - 1/r_{13})(r_{02} - 1/r_{02}) + k_{01}k_{23} - k_{03}k_{12}, \\ c &= k_{01}/r_{02} + r_{13}k_{23} - k_{03}r_{13}/r_{02} - k_{12}, \\ d &= k_{12} - r_{02}k_{01} - r_{13}k_{23} + r_{02}r_{13}k_{03}. \end{aligned} \quad (\text{E.17})$$

In addition we introduce

$$\gamma_{kl} = \text{sign } \Re[a(x_k - x_l)], \quad k, l = 1, 2, \quad (\text{E.18})$$

and

$$\begin{aligned}
 x_{k0} &= x_k, & s_0 &= \tilde{r}_{03}, \\
 x_{k1} &= x_k/r_{13}, & s_1 &= \tilde{r}_{01}, \\
 x_{k2} &= x_k r_{02}/r_{13}, & s_2 &= \tilde{r}_{12}, \\
 x_{k3} &= x_k r_{02}, & s_3 &= \tilde{r}_{23}.
 \end{aligned} \tag{E.19}$$

as well as

$$x_{kj}^{(0)} = \lim_{\varepsilon \rightarrow 0} x_{kj} \quad \text{as} \quad r_{ij} = \lim_{\varepsilon \rightarrow 0} \tilde{r}_{ij}. \tag{E.20}$$

Finally we need

$$\tilde{\eta}(a, \tilde{b}) = \begin{cases} \eta(a, b) & \text{for } b \text{ not real,} \\ 2\pi i \left[\theta(-\Im a) \theta(-\Im \tilde{b}) - \theta(\Im a) \theta(\Im \tilde{b}) \right] & \text{for } b < 0, \\ 0 & \text{for } b > 0 \end{cases} \tag{E.21}$$

with $b = \lim_{\varepsilon \rightarrow 0} \tilde{b}$.

Then the result for real r_{02} can be written as

$$\begin{aligned}
 D_0(p_{10}, p_{20}, p_{30}, m_0, m_1, m_2, m_3) &= \frac{1}{m_1 m_2 m_3 m_4 a (x_1 - x_2)} \\
 &\left\{ \sum_{j=0}^3 \sum_{k=1}^2 (-1)^{j+k} \left[\text{Li}_2(1 + s_j x_{kj}) + \eta(-x_{kj}, s_j) \log(1 + s_j x_{kj}) \right. \right. \\
 &\quad \left. \left. + \text{Li}_2\left(1 + \frac{x_{kj}}{s_j}\right) + \eta\left(-x_{kj}, \frac{1}{s_j}\right) \log\left(1 + \frac{x_{kj}}{s_j}\right) \right] \right. \\
 &+ \sum_{k=1}^2 (-1)^{k+1} \left[\tilde{\eta}(-x_k, \tilde{r}_{02}) \left[\log(r_{02} x_k) + \log\left(Q\left(\frac{1}{x_k^{(0)}}, 0, 0, 1\right) - i\varepsilon\right) \right. \right. \\
 &\quad \left. \left. + \log\left(\frac{\overline{Q}(0, 0, 1, r_{02} x_k^{(0)})}{d} + i\varepsilon \gamma_{k,3-k} \text{sign}(r_{02} \Im \tilde{r}_{13})\right) \right] \right. \\
 &\quad \left. + \tilde{\eta}\left(-x_k, \frac{1}{\tilde{r}_{13}}\right) \left[\log\left(\frac{x_k}{r_{13}}\right) + \log\left(Q\left(\frac{r_{13}}{x_k^{(0)}}, 1, 0, 0\right) - i\varepsilon\right) \right. \right. \\
 &\quad \left. \left. + \log\left(\frac{\overline{Q}(1, 0, 0, x_k^{(0)})}{d} + i\varepsilon \gamma_{k,3-k} \text{sign}(\Im \tilde{r}_{13})\right) \right] \right. \\
 &\quad \left. - \left[\tilde{\eta}\left(-x_k, \frac{\tilde{r}_{02}}{\tilde{r}_{13}}\right) + \eta\left(\tilde{r}_{02}, \frac{1}{\tilde{r}_{13}}\right) \right] \left[\log\left(\frac{r_{02} x_k}{r_{13}}\right) + \log\left(Q\left(\frac{r_{13}}{x_k^{(0)}}, 1, 0, 0\right) - i\varepsilon\right) \right. \right. \\
 &\quad \left. \left. + \log\left(\frac{\overline{Q}(0, 0, 1, r_{02} x_k^{(0)})}{d} + i\varepsilon \gamma_{k,3-k} \text{sign}(r_{02} \Im \tilde{r}_{13})\right) \right] \right. \\
 &\quad \left. \left. + \eta\left(\tilde{r}_{02}, \frac{1}{\tilde{r}_{13}}\right) \tilde{\eta}\left(-x_k, -\frac{\tilde{r}_{02}}{\tilde{r}_{13}}\right) \right] \right\}. \tag{E.22}
 \end{aligned}$$

In the case that $|r_{ij}| = 1$ for all r_{ij} , the result reads:

$$\begin{aligned}
 D_0(p_{10}, p_{20}, p_{30}, m_0, m_1, m_2, m_3) &= \frac{1}{m_1 m_2 m_3 m_4 a(x_1 - x_2)} \\
 &\left\{ \sum_{j=0}^3 \sum_{k=1}^2 (-1)^{j+k} \left[\text{Li}_2(1 + s_j x_{kj}) + \eta(-x_{kj}, s_j) \log(1 + s_j x_{kj}) \right. \right. \\
 &\quad \left. \left. + \text{Li}_2\left(1 + \frac{x_{kj}}{s_j}\right) + \eta\left(-x_{kj}, \frac{1}{s_j}\right) \log\left(1 + \frac{x_{kj}}{s_j}\right) \right] \right. \\
 &+ \sum_{k=1}^2 (-1)^{k+1} \left[\eta\left(-x_k, \frac{1}{r_{13}}\right) \left[\log\left(\frac{r_{13}}{x_k^{(0)}} P\left(1, \frac{x_k^{(0)}}{r_{13}}, 0, 0\right) - \frac{x_k^{(0)}}{r_{13}} \varepsilon b \gamma_{k,3-k}\right) + \log\left(\frac{x_k^{(0)}}{r_{13}}\right) \right] \right. \\
 &\quad \left. + \eta\left(-x_k, r_{02}\right) \left[\log\left(\frac{1}{r_{02} x_k^{(0)}} P\left(0, 0, 1, r_{02} x_k^{(0)}\right) - r_{02} x_k^{(0)} \varepsilon b \gamma_{k,3-k}\right) + \log\left(r_{02} x_k^{(0)}\right) \right] \right. \\
 &\quad \left. - \left[\eta\left(-x_k, \frac{r_{02}}{r_{13}}\right) + \eta\left(r_{02}, \frac{1}{r_{13}}\right) \right] \left[\log\left(\frac{r_{13}}{r_{02} x_k^{(0)}} P\left(0, 1, \frac{r_{02} x_k^{(0)}}{r_{13}}, 0\right) - \frac{r_{02} x_k^{(0)}}{r_{13}} \varepsilon b \gamma_{k,3-k}\right) \right. \right. \\
 &\quad \left. \left. + \log\left(\frac{r_{02} x_k^{(0)}}{r_{13}}\right) \right] + \left(1 - \gamma_{k,3-k} \text{sign}(b)\right) \eta\left(-x_k, -\frac{r_{02}}{r_{13}}\right) \eta\left(r_{02}, \frac{1}{r_{13}}\right) \right] \left. \right\}.
 \end{aligned}$$

ε is understood as infinitesimally small.

E.5 UV-divergent parts of tensor integrals

For practical calculations it is useful to know the UV-divergent parts of the tensor integrals explicitly. We give directly the products of $D - 4$ with all divergent one-loop tensor coefficient integrals appearing in renormalizable theories up to terms of the order $O(D - 4)$

$$\begin{aligned}
 (D - 4) A_0(m) &= -2m^2, \\
 (D - 4) B_0(p_{10}, m_0, m_1) &= -2, \\
 (D - 4) B_1(p_{10}, m_0, m_1) &= 1, \\
 (D - 4) B_{00}(p_{10}, m_0, m_1) &= \frac{1}{6}(p_{10}^2 - 3m_0^2 - 3m_1^2), \\
 (D - 4) B_{11}(p_{10}, m_0, m_1) &= -\frac{2}{3}, \\
 (D - 4) C_{00}(p_{10}, p_{20}, m_0, m_1, m_2) &= -\frac{1}{2}, \\
 (D - 4) C_{00i}(p_{10}, p_{20}, m_0, m_1, m_2) &= \frac{1}{6}, \\
 (D - 4) D_{0000}(p_{10}, p_{20}, p_{30}, m_0, m_1, m_2, m_3) &= -\frac{1}{12}.
 \end{aligned} \tag{E.23}$$

All other scalar coefficients defined in (??) and (??) are UV-finite.

Appendix A brief introduction to cosmology

We present in this section a very brief introduction to cosmology, useful to understand the results of Sec. 11.

F.1 Homogeneity, isotropy and expansion

We all know how the universe looks like locally, i.e. in the solar system. One surprising feature is that, when considering scales larger than about 100 Mpc, experimentally our universe looks *homogeneous* and *isotropic*. Such result can be partially inferred from galaxy surveys like the SLOAN digital telescope, although the measurement is clearly very difficult. Why do we say then that the universe *is* homogeneous? The reason is two-fold: (i) the predictions from the theory in which homogeneity is built-up from the very beginning are very successful, and (ii) homogeneity is related to isotropy in a non-trivial way. Isotropy states that as we stand in any point, the universe will look the same in any direction. It is very hard to construct a model of universe in which we have isotropy but not homogeneity, at least as long as we imagine isotropy not to be only a property valid locally around us and we suppose that all observers see isotropy. Moreover, we have a very strong experimental evidence for isotropy, given by the Cosmic Microwave Background (CMB) map. This map measures radiation coming from all possible directions, with a wavelength of the order of μm and an almost perfect black-body spectrum. Once the so-called dipole radiation is subtracted ¹, the CMB map shows that the background radiation we measure is isotropic to a very good degree, with photons with an energy such that their temperature is

$$T_{CMB} = 2.7 \text{ K.} \tag{F.1}$$

The measured violation of isotropy is of the order of $\Delta T/T \sim 10^{-5}$, i.e. extremely small. This allows us to conclude that the universe is indeed homogeneous and isotropic, at least on large scales. Clearly it is not so at small scales, at which we measure structures like planets, stars, galaxies and clusters. In fact, all the structures we see today are due to the small violations of isotropy measured in the CMB map. It is believed that such deviations are generated by quantum effects in the primordial stage of evolution called inflation.

A second piece of information that we must be aware of is the fact that *the universe*

¹Dipole radiation is due to the Earth motion.

is expanding. All galaxies we observe are receding from us, with an expansion which is accelerating.

More quantitatively, to describe the universe we will use General Relativity and implement the hypothesis of homogeneity and isotropy in the metric. Such hypothesis turns out to be quite restrictive, allowing only for the Friedmann-(Lemaitre)-Robertson-Walker (FRW) metric:

$$ds^2 = dt^2 - a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right) \equiv dt^2 - a^2(t)\gamma_{ij}dr^i dr^j. \quad (\text{F.2})$$

The temporal part of the metric is the same as in Minkowski space. As for the spatial part, the parameter k can take values $\{-1, 0, +1\}$ in units of the space curvature according to the type of universe described (open, flat or closed). The “radius” r is called “comoving radius”, while $a(t)$ is called *scale factor*. Conventionally, its value today is fixed to $a_0 = 1$. The form of the FRW metric guarantees homogeneity and isotropy. Notice that the comoving coordinates are constant, i.e. they do not depend on time. What depends on time is the scale factor and, through it, the physical distances R :

$$R(t) = a(t)r. \quad (\text{F.3})$$

Locally we can measure the velocity with which nearby galaxies are receding from us, observing

$$\dot{R} \simeq H_0 R, \quad H_0 \simeq 70 \frac{\text{km/s}}{\text{Mpc}}. \quad (\text{F.4})$$

Comparing with Eq. (F.3) we immediately deduce

$$\dot{R} = \dot{a}r = \frac{\dot{a}}{a}R. \quad (\text{F.5})$$

We are thus lead to consider the Hubble parameter

$$H(t) = \frac{\dot{a}(t)}{a(t)}, \quad (\text{F.6})$$

whose value today is H_0 . We will always use the subscript 0 to denote quantities computed today.

F.2 Consequences of expansion

Expansion plays a fundamental role in explaining current observations. Consider for instance the CMB map that, as already mentioned, shows an almost perfect black body spectrum. From statistical mechanics we know that a black body spectrum is produced once the system achieves a maximum entropy state, i.e. after the photons have scattered many times. How many scatterings does a typical CMB photon undergo

during his history? We can estimate such number computing the *optical depth*

$$\tau = \frac{ct_{universe}}{\ell_{mfp}}, \quad (\text{F.7})$$

where $t_{universe}$ is the approximate age of the universe and ℓ_{mfp} is the photon mean free path, i.e. the average distance that a photon can travel between two scatterings. The oldest objects we see in the universe date back to $t \sim 13$ Gyr, so that we will take $t_{universe} \sim 10$ Gyr. As for the mean free path, we estimate it considering photon scattering off electrons via Thomson processes:

$$\ell_{mfp} = \frac{1}{\sigma_T n_e}, \quad \sigma_T = \frac{8\pi}{3} \frac{\alpha^2}{m_e^2} \simeq 6.6 \times 10^{-25} \text{ cm}^2, \quad n_e \simeq 2 \times 10^{-7} \frac{e}{\text{cm}^3}. \quad (\text{F.8})$$

The Thomson cross section can be easily computed from a tree level diagram for the $\gamma e \rightarrow \gamma e$ process. The quantity n_e is the local electron density, taken here as a proxy of the total electron density in the universe). The result is that the typical CMB photon underwent on average just very few scattering $\tau \sim 10^{-3}$. How do we then explain the almost perfect black body spectrum we observe? The way out is to consider expansion: since the universe is expanding, the electron density grows larger as we go backwards in time. The correct estimate for the optical depth is

$$\tau = \int_0^{t_{universe}} dt c \sigma_T n_e(t), \quad (\text{F.9})$$

and it is clear that it can be much larger than our naive estimate. Another very interesting consequence of expansion is that the photon energy is redshifted. To see this let us consider the free photon action in the FRW metric:

$$S = -\frac{1}{4} \int d^4r \sqrt{-g} g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}. \quad (\text{F.10})$$

Since realistic photons have wavelengths much smaller than any possible universe curvature, it is a good approximation to set $k = 0$ in the FRW metric, i.e. we take $\gamma_{ij} = \delta_{ij}$ in Eq. (F.2).² The length element with this approximation can be simplified by introducing the conformal time

$$dt = a(t) d\eta, \quad (\text{F.11})$$

in terms of which we can write

$$ds^2 = a^2(\eta) (d\eta^2 - \mathbf{dr} \cdot \mathbf{dr}). \quad (\text{F.12})$$

We obtain a length element proportional to the Minkowski one apart from an overall scale factor: $g_{\mu\nu} = a^2(\eta) \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric. Its inverse and determinant are given by

$$g^{\mu\nu} = \frac{1}{a^2(\eta)} \eta^{\mu\nu}, \quad \sqrt{-g} = a^4(\eta), \quad (\text{F.13})$$

²Experimental evidence, mainly from the CMB, points in any case towards a spatially flat universe.

in such a way that the action in the (η, \mathbf{r}) coordinates simply reads

$$S = -\frac{1}{4} \int d^4r \sqrt{-g} \eta^{\mu\nu} \eta^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}. \quad (\text{F.14})$$

We obtain the action of a photon in Minkowski space. The equation of motion in the (η, \mathbf{r}) coordinates is thus simply solved by a plane wave solution

$$A_\mu^{(\lambda)} = e_\mu^{(\lambda)} e^{-i(k\eta - \mathbf{k}\cdot\mathbf{r})}, \quad (\text{F.15})$$

where \mathbf{k} is the *conformal or comoving momentum* of the photon. The physical momentum can be inferred from the fact that

$$\mathbf{k} \cdot \mathbf{r} = \mathbf{k} \cdot \frac{\mathbf{R}}{a(t)} = \frac{\mathbf{k}}{a(t)} \cdot \mathbf{R} = \mathbf{p} \cdot \mathbf{R} \quad \Rightarrow \quad \mathbf{p} = \frac{\mathbf{k}}{a(t)}, \quad E = \frac{k}{a(t)}. \quad (\text{F.16})$$

The photon momentum and energy diminish with time, since they are inversely proportional to the scale factor. We now define the *redshift* $z(t)$ as

$$a(t) \equiv \frac{1}{1 + z(t)}. \quad (\text{F.17})$$

Since the scale factor today has a value $a_0 = 1$, the redshift is such that $z_0 = 0$. The earlier in the cosmological history, the larger the value of the redshift. In terms of the redshift the photon energy can be written as

$$E(t') = k (1 + z(t')) = \frac{1 + z(t')}{1 + z(t)} E(t). \quad (\text{F.18})$$

Take now $t' = t_{emis}$ as the time of photon emission and $t = t_0$ as the time at which we measure the signal of such photon:

$$E(t_{emis}) = (1 + z(t_{emis})) E_{obs}. \quad (\text{F.19})$$

The energy at emission was larger than the energy at observation, i.e. the photon energy is redshifted in its propagation. As we are going to see in Sec. F.4, this has important effects on the photon distribution.

F.3 Dynamics of $a(t)$ and Friedmann equations

Let us now study how the scale factor a depends on time (or, equivalently, on redshift). The dynamics of the universe is found solving the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \quad (\text{F.20})$$

together with the covariant conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$. The hypothesis of homogeneity and isotropy must be valid also for the energy-momentum tensor of our universe. Isotropy implies that the off-diagonal terms must vanish (they would introduce preferred directions in spacetime), reducing the form of the $T_{\mu\nu}$ tensor to

$$T_{00} = \rho, \quad T_{ij} = -pa^2(t)\gamma_{ij}. \quad (\text{F.21})$$

ρ and p are the energy density and effective pressure, respectively. Homogeneity furthermore implies that $\rho = \rho(t)$ and $p = p(t)$ can only depend on time. This is the same energy-momentum tensor we would obtain for a perfect fluid. Inserting it in the Einstein and covariant conservation equations we obtain the so-called *Friedmann equations*

$$H^2 = \frac{8\pi G}{2}\rho - \frac{k}{a^2}, \quad \dot{\rho} = -3H(\rho + p). \quad (\text{F.22})$$

In what follows we will always set $k = 0$, since it is at least an excellent approximation for our universe. To close the system of equations we also need the *equation of state* $p = w\rho$. We will be interested in the following fluids: *dust* ($w = 0$), *radiation* ($w = 1/3$) and *cosmological constant* ($w = -1$). The general solution to

$$\dot{\rho} = -3(1 + w)H \quad (\text{F.23})$$

is

$$\rho(t) = \begin{cases} \frac{\text{const}}{a^{3(1+w)}} & w \neq -1 \\ \text{const} & w = -1 \end{cases} \quad \text{or} \quad \begin{cases} \rho_0 (1+z)^{3(1+w)} & w \neq -1 \\ \rho_0 & w = -1 \end{cases} \quad (\text{F.24})$$

As for the solution of the first Friedmann equation, it is convenient to introduce the *critical density*

$$\rho_c = \frac{3H_0^2}{8\pi G} \quad (\text{F.25})$$

and, for each component of the universe, $\Omega_i = \rho_i/\rho_c$. Observations (CMB, baryon acoustic oscillations) today point to

$$\begin{aligned} \Omega_b &\simeq 0.04, \\ \Omega_M &\simeq 6 \times \Omega_b \simeq 0.3, \\ \Omega_\Lambda &\simeq 0.7, \\ \Omega_R &\simeq 10^{-5}, \end{aligned} \quad (\text{F.26})$$

where b stands for baryons (i.e. normal matter), M stands for the total amount of matter, Λ stands for the cosmological constant and R for radiation. As we can see, the total amount of matter is about a factor of $5 \div 6$ larger than the total amount of baryons. This is one of the evidences for the existence of Dark Matter. The first Friedmann equation can now be written as

$$H^2 = \frac{8\pi G}{3} [\rho_\Lambda + \rho_M(0)(1+z)^3 + \rho_R(0)(1+z)^4] \quad (\text{F.27})$$

or

$$H = H_0 \sqrt{\Omega_\Lambda + \Omega_M(1+z)^3 + \Omega_R(1+z)^4}. \quad (\text{F.28})$$

Depending on the value of z , some notable moments in the cosmological history can already be identified:

- Λ domination: when did the cosmological constant contribution start to dominate?

We can estimate when this happened imposing

$$\Omega_\Lambda \simeq \Omega_M(1 + z_\Lambda)^3 \quad \Rightarrow \quad z_\Lambda \simeq 0.3. \quad (\text{F.29})$$

The cosmological constant domination is thus a pretty recent cosmological event.

For $z \simeq 0.3$ the matter contribution is the dominant one in the universe;

- Radiation domination: moving even backwards in time we arrive at a moment in which radiation dominates over the other components. This *equality* moment can be estimated as

$$\Omega_M(1 + z_{eq})^3 = \Omega_R(1 + z_{eq})^4 \quad \Rightarrow \quad z_{eq} \simeq 3.4 \times 10^3. \quad (\text{F.30})$$

Other notable moments in the history of the universe are *recombination*, when H atoms are formed ($z_{REC} \simeq 1400$) and *primordial nucleosynthesis* (or Big Bang Nucleosynthesis, BBN), when nuclei are formed ($z_{BBN} \sim 10^9$).

F.4 Distribution functions for free particles

Let us consider the distribution function $f(\mathbf{R}, \mathbf{p})$ of *free particles*, which depends on the physical coordinate \mathbf{R} and the physical momentum \mathbf{p} . For a FRW universe there cannot be any position dependence, and the only dependence on the momentum must be through its modulus, $p = |\mathbf{p}|$. Since the distribution function is defined as $f(p) = d^3n/d^3p$ and we know that the physical momentum scales as $pa = \text{const}$ and the number density n must fall as a^{-3} because of the universe expansion, we conclude that the distribution function must be constant in time. This allows us to reach two important conclusions: (i) the form of the distribution function does not change with time and (ii) the only possible argument of f is $p(t)a(t) = \text{const}$.

The previous observations allow to explain why the CMB photons have a Planckian distribution: if in the past the photons were in thermal equilibrium, then after they exit from the equilibrium state (at *decoupling*) their distribution function maintains the same functional form. The CMB observation thus tells us that in the past the universe passed an equilibrium stage. We look now at the Planckian distribution at the moment of decoupling:

$$f(p) = \frac{1}{(2\pi)^3} \frac{1}{e^{E_D/T_D} - 1}. \quad (\text{F.31})$$

Remembering that we must have $f(ap) = f(aE)$, we conclude that the temperature must itself depend on time, and we must have

$$\frac{E_D}{T_D} = \frac{a_D E_D}{a_D T_D} = \frac{a(t) E(t)}{a_D T_D}. \quad (\text{F.32})$$

The argument of the distribution takes the usual form provided we define

$$T(t) = \frac{a_D T_D}{a(t)}. \quad (\text{F.33})$$

This confirms our previous conclusion: the temperature grows going backwards in time, and thus the universe can have been in a thermal equilibrium state.

The situation is different for massive particles. The dependence $f(ap)$ is still the same, but the actual form

F.5 Equilibrium and non-equilibrium processes

As we have seen, CMB observations suggest that the universe passed through a period of thermodynamical equilibrium in his history. Let us point out that, in a sense, thermal equilibrium is “boring”, since nothing is happening (the interactions are so fast that everything behaves basically in the same way). Interesting physics happens when we have departures from equilibrium. In this section we will first define some interesting quantities useful to describe equilibrium states (number and energy density, pressure) and then discuss how to describe the departure from the equilibrium state introducing the Boltzmann equation.

F.5.1 Equilibrium

The central object we will use to describe the equilibrium state is the *equilibrium phase space distribution*

$$f(p) = \frac{1}{e^{(E-\mu)/T} \mp 1}. \quad (\text{F.34})$$

In the previous expression the minus sign applies to bosons (Bose-Einstein distribution), the plus sign to fermions (Fermi-Dirac distribution) and μ is the chemical potential, defined as

$$\mu = \sum_i \mu_i Q_i, \quad (Q_i = \text{conserved quantity}). \quad (\text{F.35})$$

In the limit $E - \mu \gg T$ the ∓ 1 term in the denominator can be discarded and we obtain the Maxwell-Boltzmann distribution

$$f_{MB}(p) = e^{-(E-\mu)/T}. \quad (\text{F.36})$$

The phase space distribution can be used to define three important quantities: the number density n , the energy density ρ and the pressure p of the system:

- Number density of a species i :

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} f(p) = \begin{cases} g_i \frac{\zeta(3)}{\pi^2} T^3 & \text{(relativistic boson)} \\ \frac{3}{4} g_i \frac{\zeta(3)}{\pi^2} T^3 & \text{(relativistic fermion)} \\ g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-(m_i - \mu_i)/T} & \text{(non - relativistic particle)} \end{cases} \quad (\text{F.37})$$

In the previous expression g_i denotes the number of degrees of freedom (for instance, $g_i = 2$ for a Weyl fermion, $g_i = 4$ for a Dirac fermion, $g_i = 2$ for a massless vector etc.);

- Energy density of a species i :

$$\rho_i = g_i \int \frac{d^3p}{(2\pi)^3} E(p) f(p) = \begin{cases} g_i \frac{\pi^2}{30} T^4 & \text{(relativistic boson)} \\ \frac{7}{8} g_i \frac{\pi^2}{30} T^4 & \text{(relativistic fermion)} \\ m_i n_i + \frac{3}{2} n_i T & \text{(non - relativistic particle)} \end{cases} \quad (\text{F.38})$$

A useful definition is the one of “total energy density in radiation”:

$$\rho_R = \frac{\pi^2}{30} g_* T^4, \quad g_* = \sum_b g_b + \frac{7}{8} \sum_f g_f. \quad (\text{F.39})$$

Combining $\rho_R \sim T^4$ with $\rho_R \sim a^{-4}$ we recover $T \sim a^{-1}$, which is the redshift relation already discussed;

- Pressure of a species i :

$$p_i = g_i \int_{p_z > 0} \frac{d^3p}{(2\pi)^3} \frac{2p_z^2}{E(p)^2} f(p) = \begin{cases} \frac{\rho_i}{3} & \text{(relativistic particle)} \\ n_i \ll \rho_i T & \text{(non - relativistic particle)} \end{cases} \quad (\text{F.40})$$

This definition is easily justified: consider a set of particles with velocity $v_z = p_z/E(p)$ moving towards a wall. The total pressure on the wall will be

$$\begin{aligned} p &= \frac{\Delta \text{momentum}}{dt dS} \\ &= \frac{2p_z \times (\# \text{incident particles})}{dt dS} \\ &= \frac{2p_z \times (f(p) d^3p dS v_z dt)}{dt dS} \\ &= \frac{2p_z^2}{E(p)} f(p) d^3p. \end{aligned} \quad (\text{F.41})$$

This justify the equation used above. We can also rewrite this expression in an equivalent form using the isotropy of momentum, that implies $p_z^2 = p^2/3$. Moreover, the symmetry of the integrand under a $p_z \rightarrow -p_z$ reflection implies that

$$\int_{p_z > 0} d^3p = \frac{1}{2} \int d^3p.$$

Putting all together, an alternative definition of the pressure is

$$p_i = g_i \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E(p)^2} f(p). \quad (\text{F.42})$$

From the results above we immediately conclude that the equation of state is $w = 1/3$ for radiation and $w \simeq 0$ for non-relativistic matter.

When talking about thermal equilibrium we are actually talking about two different types of equilibrium:

- *kinetic equilibrium*: fast number-conserving interactions force the phase space distribution to be like in Eq. (F.34). An example is $\gamma e \leftrightarrow \gamma e$;
- *chemical equilibrium*: fast number-changing interactions of the type $\gamma\gamma \leftrightarrow e^-e^+$ force the chemical potentials of the two sides of the reaction to be equal. In general, given a reaction $A_1 + A_2 + \dots \leftrightarrow B_1 + B_2 + \dots$, chemical equilibrium implies $\mu_{A_1} + \mu_{A_2} + \dots = \mu_{B_1} + \mu_{B_2} + \dots$. This has important consequences. First of all, bremsstrahlung processes like $ee \rightarrow ee\gamma$ force $\mu_\gamma = 0$. Using now the $\gamma\gamma \leftrightarrow e^+e^-$ we immediately conclude that $\mu_{e^-} = \mu_{e^+}$, i.e. the chemical potentials of an antiparticle is the opposite of the one of the corresponding particle.

F.5.2 Out-of-equilibrium processes

As we have seen in the previous section, equilibrium is associated with fast interactions. This implies that non-equilibrium processes are somehow associated with reactions becoming slow. To describe the time evolution of a system that is going out of equilibrium we use the *Boltzmann equation*, a differential equation describing the time variation of the number density of a certain species. We now derive this equation.

Let us start by considering a reaction

$$1 + 2 \leftrightarrow 3 + 4. \quad (\text{F.43})$$

We are interesting in computing the time variation \dot{N}_1 , where N_1 is the number of particles of type 1 contained in a physical volume V . Notice that we can always write

$$\dot{N}_1 = \frac{\text{change due to reactions}}{\text{time}}, \quad (\text{F.44})$$

where the right hand side will depend on the type of interactions. Before computing this term, we observe that the physical volume V can always be written as $V = a^3 V_{\text{comoving}}$, with the comoving volume constant by definition. We can thus explicitly compute \dot{N}_1 as

follows:

$$\begin{aligned}
 \dot{N}_1 &= \frac{d}{dt}(n_1 V) = \frac{d}{dt}(n_1 a^3 V_{\text{comoving}}) \\
 &= \dot{n}_1 V + 3\dot{a} a^2 n_1 V_{\text{comoving}} \\
 &= (\dot{n}_1 + 3H n_1) V.
 \end{aligned} \tag{F.45}$$

These terms represent the variation in N_1 due to the expansion of the universe. We now move to the computation of the collision term. We have

$$\frac{\text{change due to reactions}}{\text{time}} = -N_{12} \dot{P}_{12 \rightarrow 34} + N_{34} \dot{P}_{34 \rightarrow 12}, \tag{F.46}$$

where N_{ij} is the number of ij pair in the physical volume V and $\dot{P}_{ij \rightarrow km}$ is the probability per unit time for the reaction $ij \rightarrow km$ to happen. Supposing that there is no correlation between the number of different species of particles present in the physical volume V , the number of pairs can be written as

$$N_{ij} = \int \frac{d^3 p_i}{(2\pi)^3} \frac{d^3 p_j}{(2\pi)^3} f_i f_j V^2. \tag{F.47}$$

The probability per unit time for the reaction $ij \rightarrow km$ to happen is instead given by the usual formula involving the S-matrix element ³

$$\dot{P}_{ij \rightarrow km} = \int \frac{d^3 p_k}{(2\pi)^3/V} \frac{d^3 p_m}{(2\pi)^3/V} \frac{d}{dt} \frac{|\langle km|S|ij\rangle|^2}{\langle ij|ij\rangle \langle km|km\rangle}. \tag{F.48}$$

Remembering now that

$$\frac{d}{dt} |\langle km|S|ij\rangle|^2 = (2\pi)^4 V \delta^4(p_i + p_j - p_k - p_m) |M_{ij \rightarrow km}|^2 \tag{F.49}$$

and that, with box quantization, the state normalization is

$$\langle i|i\rangle = 2E_i V, \tag{F.50}$$

we can rewrite the term $N_{ij} \dot{P}_{ij \rightarrow km}$ as

$$\begin{aligned}
 N_{ij} \dot{P}_{ij \rightarrow km} &= \int \frac{d^3 p_i}{(2\pi)^3/V} \frac{d^3 p_j}{(2\pi)^3/V} \frac{d^3 p_k}{(2\pi)^3/V} \frac{d^3 p_m}{(2\pi)^3/V} f_i f_j \frac{(2\pi)^4 V \delta^4(p_{in} - p_{fin}) |M_{ij \rightarrow km}|^2}{(2E_i V)(2E_j V)(2E_k V)(2E_m V)} \\
 &= V \int d\Pi_i d\Pi_j d\Pi_k d\Pi_m f_i f_j (2\pi)^4 \delta^4(p_{in} - p_{fin}) |M_{ij \rightarrow km}|^2,
 \end{aligned} \tag{F.51}$$

where we have introduced the usual relativistic phase space measure

$$d\Pi_i = \frac{d^3 p_i}{(2\pi)^3 2E_i}.$$

³More correctly, we should include factors of $(1 \pm f_k)(1 \pm f_m)$ inside the integral of the following formula, where the minus sign applies to fermions and the plus sign to bosons. We however suppose that the system is sufficiently diluted to neglect such terms.

We are finally in a position of writing the Boltzmann equation for the $12 \leftrightarrow 34$ interaction:

$$\begin{aligned} \dot{n}_1 + 3Hn_1 &= - \int d\Pi_1 d\Pi_2 d\Pi_3 d\Pi_4 (2\pi)^4 \delta^4(p_{in} - p_{fin}) (f_1 f_2 |M_{12 \rightarrow 34}|^2 - f_3 f_4 |M_{34 \rightarrow 12}|^2) \\ &= - \int d\Pi_1 d\Pi_2 (f_1 f_2 \sigma_{12 \rightarrow 34} v - f_3 f_4 \sigma_{34 \rightarrow 12} v), \end{aligned} \quad (\text{F.52})$$

where in the last step we used the definition of cross section. This equation can be generalized to arbitrary reactions (for instance decays or $3 \rightarrow 2$ processes).

The rest of this section will be devoted to the simplification of the Boltzmann equation. We will make the following assumptions (that must be checked case by case in explicit cases):

- We will assume time reversal symmetry:

$$M_{12 \rightarrow 34} = M_{34 \rightarrow 12};$$

- We will suppose the 3, 4 particles to be in kinetic equilibrium:

$$f_3 f_4 = f_3^{eq} f_4^{eq};$$

- We will approximate the equilibrium distributions as Maxwell-Boltzmann:

$$f_3^{eq} f_4^{eq} = e^{-(E_3+E_4)/T} = e^{-(E_1+E_2)/T} = f_1^{eq} f_2^{eq},$$

where we have used energy conservation (which is enforced by the delta function in the integral);

- We will suppose

$$f_{1,2} \propto f_{1,2}^{eq} \quad \Rightarrow \quad f_{1,2} = \frac{n_{1,2}}{n_{1,2}^{eq}} f_{1,2}^{eq}.$$

Putting together these hypothesis we can simplify the collision integral into the following expression:

$$\dot{n}_1 + 3Hn_1 = - (n_1 n_2 - n_1^{eq} n_2^{eq}) \underbrace{\frac{\int d\Pi_1 d\Pi_2 f_1^{eq} f_2^{eq} \sigma v}{n_1^{eq} n_2^{eq}}}_{\langle \sigma v \rangle}, \quad (\text{F.53})$$

where we have defined the thermally averaged cross section $\langle \sigma v \rangle$. This factor is completely fixed by the theory under consideration.

In order to solve this differential equation it is convenient to introduce the yield

$$Y \equiv \frac{n}{s}, \quad (\text{F.54})$$

where in the denominator we have introduced the *entropy density*

$$s \equiv \frac{\rho + p}{T} = \frac{2\pi^2}{45} g_{*S} T^3, \quad g_{*S} = \sum_b g_b \left(\frac{T_b}{T}\right)^3 + \frac{7}{8} \sum_f g_f \left(\frac{T_f}{T}\right)^3. \quad (\text{F.55})$$

The concept of entropy density is useful because, as long as we neglect the temperature

dependence of g_{*S} , we have

$$sa^3 = \text{const}, \quad (\text{F.56})$$

i.e. there is conservation of entropy in a comoving volume. This implies that $\dot{s} = -3Hs$.

Using the yield we can write

$$\dot{n} + 3Hn = s\dot{Y}. \quad (\text{F.57})$$

We now change variable to $x \equiv m/T$, where m is the mass of the particle on which number density evolution we are interested to:

$$\frac{d}{dt} = \frac{dx}{dt} \frac{d}{dx} = -\frac{m}{T^2} \dot{T} \frac{d}{dx} = -x \frac{\dot{T}}{T} \frac{d}{dx}. \quad (\text{F.58})$$

To compute \dot{T} we suppose that the evolution happens during the radiation dominated era in which $\rho_R \propto T^4$. Deriving with respect to time we obtain

$$\dot{\rho}_R = 4\rho_R \frac{\dot{T}}{T} = -3H\rho_R \frac{4}{3} \Rightarrow \dot{T} = -HT. \quad (\text{F.59})$$

In the last step we have used the Friedmann equation. Putting all together we obtain

$$\frac{d}{dt} = xH \frac{d}{dx}, \quad (\text{F.60})$$

and the Boltzmann equation written in terms of the yield Y_1 is

$$\frac{dY_1}{dx} = -\frac{\langle\sigma v\rangle s}{xH} (Y_1 Y_2 - Y_1^{eq} Y_2^{eq}). \quad (\text{F.61})$$

We now specialize to the case in which $2\bar{1}$ i.e. we consider the annihilation process $1\bar{1} \leftrightarrow 34$. We will also suppose that no asymmetry is present, i.e. $n_1 = \bar{n}_1$. We obtain

$$\frac{dY}{dx} = -\frac{\langle\sigma v\rangle s}{xH} (Y^2 - [Y_{eq}]^2). \quad (\text{F.62})$$

The equation can be further rearranged writing $s = n_{eq}/Y_{eq}$:

$$\frac{1}{Y_{eq}} \frac{dY}{dx} = \frac{n_{eq}\langle\sigma v\rangle}{xH} \left(\frac{Y^2}{Y_{eq}^2} - 1 \right). \quad (\text{F.63})$$

The last expression not only motivates the definition of the *thermal rate*

$$\Gamma \equiv n_{eq}\langle\sigma v\rangle, \quad (\text{F.64})$$

but also makes clear that the fundamental quantity to compute the time evolution of the yield is the ratio Γ/H . More specifically, we see that

- when $\Gamma/H \ll 1$ we have approximately $dY/dx \simeq 0$, i.e. there is no longer evolution in the yield, i.e. the species is no longer in equilibrium. This means that $\dot{n}/n \simeq \dot{s}/s = -3H$, i.e. $n \sim a^{-3}$;
- when $\Gamma \gg H$ we have $Y \simeq Y_{eq}$. This can be intuitively understood observing that, in the limit of very large Γ/H , the left hand side must remain finite, forcing the term in brackets to be small. A more thorough derivation can be found in Phys.Rev.D 32 (1985) 3261.

The transition between the two regimes happens when $\Gamma \simeq H$. This moment defines the so-called *freeze-out* of the species under consideration. When freeze-out happens, the species decouples from the thermal bath and evolves independently from it.

Let us now estimate when neutrinos decouple from the thermal bath. We will not give a precise treatment but just look at an order-of-magnitude estimate. We need to compute the thermal rate Γ . Since neutrinos are very light, we can take the number density of a relativistic particle, $n \sim T^3$. The typical reactions that keep the neutrinos in contact with electrons are mediated by the W and Z boson, and we can thus estimate $\sigma v \sim G_F^2$ and

$$\langle \sigma v \rangle \sim G_F^2 T^2, \quad (\text{F.65})$$

where the factor of T^2 must be inserted by dimensional analysis. The Hubble parameter during radiation domination can be approximated as

$$H \sim \frac{T^2}{M_{PL}}, \quad (\text{F.66})$$

where we have defined the reduced Planck mass $M_{PL}^{-2} = 8\pi G_N/3$. We can thus estimate the freeze-out temperature as

$$G_F^2 T_{FO}^5 \sim \frac{T_{FO}^2}{M_{PL}} \Rightarrow T_{FO} \sim 2 \text{ MeV}. \quad (\text{F.67})$$

Neutrino decoupling thus happens right after BBN. After freeze-out neutrinos are decoupled from the thermal bath and form the so-called ν CB, neutrino cosmic background. It is interesting to compute also the neutrino temperature after decoupling. We will show why it does not coincide the photon temperature. At freeze-out neutrinos and photons have the same temperature, since they were just in thermal equilibrium with each other. After decoupling, both T_γ and T_ν scale as a^{-1} . At temperature of the order $T \sim m_e$ the electrons become non-relativistic and exit from the thermal bath. When this happens, the entropy in e^\pm pairs is transferred to the photons, increasing their temperature. We can estimate this change in T_γ in the following way: for $T \gtrsim m_e$ the particles in thermal equilibrium include photons ($g = 2$) and electrons ($g = 4$), producing $g_{*S} = 11/2$. For $T \ll m_e$ only the photons are in equilibrium, giving $g_{*S} = 2$. Entropy conservation per comoving volume requires

$$g_{*S} a^3 T^3 = \text{const}. \quad (\text{F.68})$$

Computing this quantity at the moment of e^\pm annihilation we obtain

$$\frac{11}{2} T_{>}^3 = 2 T_{<}^3, \quad (\text{F.69})$$

where $T_{>}$ and $T_{<}$ denote the photon temperature before and after annihilation. We thus

see that

$$T_{<} = \left(\frac{11}{4}\right)^{1/3} T_{>} \quad \Rightarrow \quad T_{\nu} = \left(\frac{4}{11}\right)^{1/3} T_{\gamma}. \quad (\text{F.70})$$

In the last step we have identified $T_{>} = T_{\nu}$ (because neutrinos are not affected by the e^{\pm} annihilation) and $T_{<} = T_{\gamma}$, the photon temperature after annihilation.

We conclude this chapter with a remark about the Boltzmann equation in Eq. (F.52).

Remembering that

$$n_1 = \int dV_{phys} f(p) \quad (\text{F.71})$$