RELATIVISTIC QUANTUM MECHANICS

1. LORENTZ GROUP

Almost always, $\hbar = 1 = c$

Lorentz tr. on a 4-vector

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

→ leaves interval invariant

$$x^2 \equiv x^{\mu} x_{\mu} = g_{\mu\nu}\, x^{\mu} x^{\nu} = x^{T} g\, x$$

matrix form

$$\left\{ \begin{array}{l} x = \{x^{\mu}\} = \begin{bmatrix} x^0 \\ \vec{x} \end{bmatrix} \\ g = \left(\begin{array}{c|c} 1 & \\ \hline & -\mathbb{1}_3 \end{array} \right) \\ x' = \Lambda x \end{array} \right.$$

$$\Rightarrow \quad x^2 = x'^2$$

$$\Downarrow$$

$$x^T g x = x^T \Lambda^T g \Lambda x \quad \Rightarrow \quad g = \Lambda^T g \Lambda$$

$$\updownarrow$$

defines

LORENTZ GROUP

$SO(1,3)$

analog of $\quad O^T O = \mathbb{1}$ for $O(N)$

$U^\dagger U = \mathbb{1}$ for $U(N)$

Properties of Λ

1) $\det(g) = \det(\Lambda^T g \Lambda) \stackrel{!}{=} (\det \Lambda)^2 \det g$

$\Longrightarrow \boxed{\det \Lambda = \pm 1}$

2) In components:

$$g_{\alpha\beta} x^\alpha x^\beta = g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha x^\beta$$

$$\Longrightarrow \boxed{g_{\alpha\beta} = g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta}$$

Take $\alpha = 0 = \beta$

$$\rightarrow g_{00} = +1 = g_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2$$

$$\rightarrow (\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3} (\Lambda^i{}_0)^2 \geq 1 \rightarrow \boxed{|\Lambda^0{}_0| \geq 1}$$

CLASSIFICATION OF LORENTZ TR.

	$\det \Lambda = +1$	$\det \Lambda = -1$
$\Lambda^0{}_0 \geq 1$	PROPER (ORTHOCRONOS) LORENTZ GROUP (can be close to $\mathbb{1}$	SPACE INVERS.
$\Lambda^0{}_0 \leq -1$	TIME INVERSION	TIME INVERS.

Proper Lorentz group $\rightarrow \Lambda$ can be close to $\mathbb{1}$

$\Downarrow$

LIE GROUP

$SO(1,3)_+$

Lie algebra of $SO(1,3)_+$

Take Λ close to $\mathbb{1}$: $\Lambda^\alpha{}_\beta \simeq \delta^\alpha{}_\beta + \hat{\omega}^\alpha{}_\beta$

Impose Lorentz condition $\Lambda^T g \Lambda = g$:

$$g_{\mu\nu} = (\delta^\alpha{}_\mu + \hat{\omega}^\alpha{}_\mu)(\delta^\beta{}_\nu + \hat{\omega}^\beta{}_\nu)\, g_{\alpha\beta}$$

$$\simeq g_{\mu\nu} + \underbrace{(\hat{\omega}_{\mu\nu} + \hat{\omega}_{\nu\mu})}_{=0}$$

PROPERTIES OF $\hat{\omega}$:

1) $\hat{\omega}_{\mu\nu} = -\hat{\omega}_{\nu\mu}$ <u>antisymmetric</u>

2) Λ real $\Rightarrow$ $(\mathbb{1} + \hat{\omega}^*) = \mathbb{1} + \hat{\omega}$

$\Rightarrow \hat{\omega}^* = \hat{\omega}$ <u>must be real</u>

EXPLICIT CONSTRUCTION $\hat{\omega}$

Most general purely real antisymmetric matrix:

$$\hat{\omega}_{\alpha\beta} = \begin{bmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ -\omega_{01} & 0 & \omega_{12} & \omega_{13} \\ -\omega_{02} & -\omega_{12} & 0 & \omega_{23} \\ -\omega_{03} & -\omega_{13} & -\omega_{23} & 0 \end{bmatrix}$$

$\Longrightarrow$ one generator for each of the independent 6 parameters

$\Downarrow$

LORENTZ GROUP HAS 6 GENERATORS

Write $\hat{\omega}_{\alpha\beta} = \frac{1}{2}\left(\omega_{\mu\nu} M^{\mu\nu}\right)_{\alpha\beta}$

antisymmetric (to get 6 independent objects)

$$\hat{\omega}^{\alpha}{}_{\beta} = g^{\alpha\mu}\,\hat{\omega}_{\mu\beta} = \begin{bmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{01} & 0 & -\omega_{12} & -\omega_{13} \\ \omega_{02} & \omega_{12} & 0 & -\omega_{23} \\ \omega_{03} & \omega_{13} & \omega_{23} & 0 \end{bmatrix} = \frac{1}{2}\left(\omega_{\mu\nu} M^{\mu\nu}\right)^{\alpha}{}_{\beta}$$

Compact way to write:

$$\left(M^{\mu\nu}\right)_{\alpha\beta} = \delta^{\mu}_{\alpha}\,\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\,\delta^{\nu}_{\alpha}$$

What are these generators?

Physically, we know that the generators should be $\begin{cases} \text{3-dim rotations} \\ \text{boosts} \end{cases}$

* From non-rel QM, $\boxed{\text{generators of rotations}}$

$$(J_1)^\alpha{}_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \; ; \; (J_2)^\alpha{}_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix};$$

$$(J_3)^\alpha{}_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

* $\boxed{\text{Boosts}}$: we know that, on a 4-vector V^μ, boost in the i^{th} direction

$$V^{0\prime} = V^0 + \frac{v_i}{c} V^i \; ; \; V^{i\prime} = V^i + \frac{v_i}{c} V^0$$

$\Rightarrow$ we immediately see that we can write

$$V'^{\mu} = \left(\delta^{\mu}{}_{\alpha} + \frac{\vec{v}}{c}\cdot\vec{K}^{\mu}{}_{\alpha}\right)V^{\alpha}$$

with

$$(K^1)^{\mu}{}_{\alpha} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ; \quad (K^2)^{\mu}{}_{\alpha} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(K^3)^{\mu}{}_{\alpha} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ 2 categories of generators:

3 symmetric $\quad \vec{K} = \{M^{01}, M^{02}, M^{03}\}$

3 antisymmetric $\quad \vec{J} = \{M^{23}, -M^{31}, M^{12}\}$

STRUCTURE CONSTANTS OF $SO(1,3)_+$

From direct computation:

$$[J_i, J_k] = \epsilon_{ikm} J_m \quad \Longleftarrow \quad \text{antisymmetric generators generate ROTATIONS}$$

$$[J_k, K_m] = \epsilon_{kmn} K_n$$

$$[K_i, K_m] = -\epsilon_{imn} J_n$$

More useful way to write the algebra $\rightarrow$ Complexify

$$J_i^{\pm} = \frac{J_i \pm i K_i}{2}$$

$$\Downarrow$$

$$\left.\begin{aligned} [J_i^+, J_k^+] &= \epsilon_{ikm} J_m^+ \\ [J_i^-, J_k^-] &= \epsilon_{ikm} J_m^- \\ [J_i^+, J_k^-] &= 0 \end{aligned}\right\} \quad \begin{matrix} \text{algebra} \\ su(2) \oplus su(2) \\ \Downarrow \text{group} \\ SU(2) \otimes SU(2) \end{matrix}$$

$\Rightarrow$ any representation of $SO(1,3)$ can be univocally determined assigning 2 semiinteger numbers (that completely determine $SU(2)$ rep.)

Since $\vec{J} = \vec{J}^{+} + \vec{J}^{-}$, the two semi-integers immediately give the spin content of the representation (just the usual sum)

$$(m_+ , m_-) \leftrightarrow |m_+ - m_-|, \dots, m_+ + m_-$$

$SO(1,3)$	$(0,0)$	$(\frac{1}{2},0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$
$SO(3)$	0	$\frac{1}{2}$	$\frac{1}{2}$	$0+1$

2 inequivalent dim = 2
representations

must be
the vector

$\Rightarrow$ at relativistic level
we have 2 independent
Spinor representations!

We know that the algebra $[J_i, J_k] = \epsilon_{ikm} J_m$ no i!
is satisfied by $\vec{J} = \left\{ \frac{i\sigma_1}{2}, -\frac{i\sigma_2}{2}, \frac{i\sigma_3}{2} \right\}$

$$\Rightarrow \begin{array}{ll} (\frac{1}{2},0) \rightarrow \vec{J}_L = \vec{J} \; ; \; \vec{K}_L = -i\vec{J} & \text{LH spinor} \\ (0,\frac{1}{2}) \rightarrow \vec{J}_R = \vec{J} \; ; \; \vec{K}_R = i\vec{J} & \text{RH spinor} \end{array}$$

$\boxed{\text{HOW DO WE MAKE CONTACT WITH 4-VECTORS?}}$

How do we translate a representation (m_+, m_-) to something 4-dim (to make contact with everything we know)?

<u>Connection through 2x2 unimodular group $SL(2,\mathbb{C})$</u>

[Borut]

Let's show that $M \in SL(2,\mathbb{C})$

$$\Downarrow$$

to $\pm M$ correspond $\Lambda(M) \in SO(1,3)_+$

and correspondence preserves multiplication

$$\pm(MN) \Rightarrow \Lambda(M)\Lambda(N) = \Lambda(MN)$$

Correspondence is two-to-one:

to both $+M$ & $-M$ corresponds

the same $\Lambda(M)$

Take

- X = any 2×2 Hermitian matrix

$\Rightarrow$ we know can be decomposed as

$$X = x^0 \mathbb{1} + \vec{x}\cdot\vec{\sigma} = x^\mu \sigma_\mu$$

where $\sigma_\mu = (\mathbb{1}, \vec{\sigma})$

- M = arbitrary 2×2 unimodular matrix

$$\det M = 1$$

Then $X' = M X M^\dagger$ again hermitian

$\Rightarrow$ must have the form $X' = x'^\mu \sigma_\mu$

Now, $\det X = \det X'$

$$\Downarrow$$

$$\underbrace{(x^0)^2 - \vec{x}^2 = (x^{0\prime})^2 - \vec{x}'^2}$$

hence, $X' = M X M^\dagger$ is a tr. that leaves the norm of x^μ invariant

$$\Downarrow$$

it's a Lorentz tr. !

Explicitly

$$X' = M x^\alpha \sigma_\alpha M^\dagger = x'^\mu \sigma_\mu = \Lambda^\mu{}_\alpha x^\alpha \sigma_\mu$$

$$\Downarrow$$

$$\boxed{M \sigma_\alpha M^\dagger = \Lambda^\mu{}_\alpha \sigma_\mu}$$

We have found

$$\Lambda \to \pm M \text{ acting on a 2-dim space}$$

$$\Downarrow$$

<u>this representation must be the spinorial!</u>

How do we make contact with what we saw before? In other words: is this $\left(\frac{1}{2}, 0\right)$ or $\left(0, \frac{1}{2}\right)$?

Remember: $\left(\frac{1}{2}, 0\right) \to \vec{J}_L = \vec{J}\,,\ \vec{K}_L = i\vec{J}$

$\left(0, \frac{1}{2}\right) \to \vec{J}_R = \vec{J}\,,\ \vec{K}_R = i\vec{J}$

with

$$\vec{J} = \left\{ \frac{i\sigma_1}{2}, -\frac{i\sigma_2}{2}, \frac{i\sigma_3}{2} \right\}$$

$\Rightarrow$ explicit LH & RH transformations read

$$\begin{cases} \psi_L' = \left[\mathbb{1} + (\vec{\alpha} - i\vec{\beta})\cdot\vec{J}\right]\psi_L \\ \psi_R' = \left[\mathbb{1} + (\vec{\alpha} + i\vec{\beta})\cdot\vec{J}\right]\psi_R \end{cases}$$

What about the M transformation?

Since $\det M = 1$, the elements of Lie group $SL(2,\mathbb{C})$ are continuously connected with $\mathbb{1}$

$\Downarrow$

$$\underbrace{M}_{\text{6 parameters!}} = \begin{bmatrix} 1 + a_{11} + ib_{11} & a_{12} + ib_{12} \\ a_{21} + ib_{21} & \underbrace{1 - a_{11} - ib_{11}}_{\text{imposing } \det = 1} \end{bmatrix}$$

$$\Rightarrow M = \mathbb{1} + (a_{11} + i b_{11})\,\sigma_3 + (a_{12} + i b_{12}) \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\frac{\sigma_1 + i\sigma_2}{2}}$$

$$+ (a_{21} + i b_{21}) \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\frac{\sigma_1 - i\sigma_2}{2}}$$

$$= \mathbb{1} + (a_{11} + i b_{11})\,\sigma_3 + \left[\frac{a_{12} + a_{21}}{2} + i\,\frac{b_{12} + b_{21}}{2}\right]\sigma_1$$

$$+ \left[\frac{a_{12} - a_{21}}{2} + i\,\frac{b_{12} - b_{21}}{2}\right] i\sigma_2$$

$$= \mathbb{1} + 2(b_{11} - i a_{11})\,J_3 + 2\left[\frac{b_{12} + b_{21}}{2} - i\,\frac{a_{12} + a_{21}}{2}\right] J_1$$

$$+ 2\left[\frac{a_{21} - a_{12}}{2} - i\,\frac{b_{12} - b_{21}}{2}\right] J_2$$

$\Longrightarrow$ we can always identify

$$\vec{\alpha} = \{ b_{12} + b_{21} , a_{21} - a_{12} , 2b_{11} \}$$

$$\vec{\beta} = \{ a_{12} + a_{21} , b_{12} - b_{21} , 2a_{11} \}$$

Then

$$M \simeq \mathbb{1} + (\vec{\alpha} - i\vec{\beta}) \cdot \vec{J}$$

$$\Downarrow$$

M can be identified with $(\frac{1}{2}, 0)$!

We define LH spinor χ_A, $A=1,2$, to transform as

$$\chi^A = M^A{}_B \, \chi^B$$

Now, at the level of 4-vectors we have invariance of scalar product

$$g_{\mu\nu} x^\mu y^\nu$$

How does this translate on spinors?

metric tensor in spinor space

$\epsilon_{AB}\,\chi^A\,\xi^B$ invariant

$\Updownarrow$

$\underbrace{\epsilon_{AB}\,M^A{}_C\,M^B{}_D}_{\text{must be } \epsilon_{CD}}\,\chi^C\,\xi^D$

must be ϵ_{CD} $\Longrightarrow$ by the determinant identity

$$M_{\alpha_1}{}^{\beta_1}M_{\alpha_2}{}^{\beta_2}\cdots M_{\alpha_N}{}^{\beta_N}\,\epsilon_{\beta_1\cdots\beta_N} = (\det M)\,\epsilon_{\alpha_1\cdots\alpha_N}$$

We now that ϵ_{AB} must be the antisymmetric symbol in 2-dim

$$\epsilon_{AB} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Invariance condition in matrix form:

$$M^T \epsilon M = \epsilon$$

$$\Big\downarrow \text{ take } * \quad (\epsilon \text{ real})$$

$$M^\dagger \epsilon M^* = \epsilon$$

$\Rightarrow$ I can construct another invariant

$$\bar{\chi}^T \epsilon \bar{\xi} \qquad \text{where } \bar{\xi} \to M^* \bar{\xi}$$

To distinguish from ξ, we introduce $\underbrace{\text{new indices}}_{\text{DOTTED}}$

$$\xi^{\dot{A}} \to (M^*)^{\dot{A}}{}_{\dot{B}}\, \xi^{\dot{B}}$$

Can we connect/identify $\bar{\xi}$ with a RH spinor?

<u>YES</u> $\longrightarrow$ explicit computation gives

$$-\epsilon\left(\mathbb{1}+(\vec{\alpha}+i\vec{\beta})\cdot\vec{J}\right)\epsilon = M^*$$

$$\Downarrow \text{ Remembering } \epsilon^2=-\mathbb{1}$$

$$\psi_R' = \underbrace{\left(\mathbb{1}+(\vec{\alpha}+i\vec{\beta})\cdot\vec{J}\right)}_{-\epsilon M^* \epsilon}\psi_R$$

$$\Downarrow$$

$$\underbrace{\epsilon\,\psi_R'}_{\bar{\xi}'} = M^* \underbrace{\epsilon\,\psi_R}_{\bar{\xi}} \quad \Bigg\| \longleftarrow$$

$\Rightarrow$ thus we find two independent spinors

$$\begin{cases} \chi_A \to M_A{}^B \chi_B \\ \bar{\chi}^{\dot{A}} \to M^{*\dot{A}}{}_{\dot{B}} \bar{\chi}^{\dot{B}} \end{cases}$$

(Van der Waerden notation)

with $\boxed{\begin{array}{l} \chi \simeq \psi_L \\ \bar{\chi} \simeq \epsilon \psi_R \end{array}}$

This finally allows us to connect

$$\left(\frac{1}{2},\frac{1}{2}\right) \longleftrightarrow x^{\mu}$$

From $X' = M X M^{\dagger}$

$$\Downarrow$$

$$X'_{A\dot{A}} = M_A{}^B \, M^*{}_{\dot{A}}{}^{\dot{B}} \, X_{B\dot{B}}$$

But $X = x^{\mu}\sigma_{\mu}$ $\Rightarrow$ σ_{μ} naturally comes
one undotted &
one dotted index

$$\Downarrow$$

it transforms in the
$\left(\frac{1}{2},\frac{1}{2}\right)$

We can also use ϵ to raise / lower indices (because it is the metric tensor in spinor space)

$$\begin{cases} \chi_A \rightarrow \chi^A = \epsilon^{AB} \chi_B \\ \epsilon^{\dot{A}\dot{B}} \epsilon^{AB} (\sigma_\mu)_{B\dot{B}} = (\mathbb{1}, -\vec{\sigma}) \equiv \bar{\sigma}_\mu \\ \vdots \\ \text{etc} \end{cases}$$

We can also ask ourselves what happens with higher dim representations.

Example : $(1,0)$

triplet of undotted indices $\Rightarrow \chi^{AB}$ symmetric

To find the corresponding tensor, we try to find combinations of σ_μ and/or $\bar{\sigma}_\mu$ which is symmetric in AB.

Result $\rightarrow$ only one possibility

$$(\sigma^{\mu\nu})_{AB} = \frac{i}{4}\left(\sigma^{\mu}{}_{A\dot{C}}\,\bar{\sigma}^{\nu\,\dot{C}D} - \sigma^{\nu}{}_{A\dot{C}}\,\bar{\sigma}^{\mu\,\dot{C}D}\right)\epsilon_{DB}$$

$\downarrow$

antisymmetric in $\mu\nu$

$$\Rightarrow \quad (1,0) \leftrightarrow \chi^{AB} \leftrightarrow \underset{\text{antisymmetric}}{A_{\mu\nu}}$$

POINCARÈ GROUP

Poincarè = Lorentz + space-time translations

$$x^\alpha \to \Lambda^\alpha{}_\beta x^\beta + a^\alpha$$

Notation: Poincarè tr. denoted with

$$(a, \Lambda)$$

Combination:

$$x \xrightarrow{(a,\Lambda)} \Lambda x + a \xrightarrow{(a',\Lambda')} \Lambda'(\Lambda x + a) + a'$$

$$\parallel$$

$$\Lambda'\Lambda x + \Lambda' a + a'$$

$$\Longrightarrow (a', \Lambda') \circ (a, \Lambda) = (\Lambda' a + a', \Lambda'\Lambda)$$

Inverse : $\underbrace{(a', \Lambda') \circ (a, \Lambda)}_{(\Lambda' a + a', \Lambda' \Lambda)} = (0, \mathbb{1})$

$$\Downarrow$$

$$\Lambda' = \Lambda^{-1} \; ; \; a' = -\Lambda' a = -\Lambda^{-1} a$$

QUANTUM MECHANICS

We seek for unitary operator

$$\hat{U}(a, \Lambda) \simeq \hat{U}(a, \mathbb{1} + \hat{\omega})$$

$$\simeq \mathbb{1} + \frac{i}{2} \omega_{\mu\nu} \hat{J}^{\mu\nu} + i\, a_\mu \hat{P}^\mu$$

Lorentz group   translations

$\Downarrow$

generator = 4-momentum

Lie algebra?

1) compute $\mathcal{U}(a,\Lambda)\,\hat{J}^{\mu\nu}\,\hat{\mathcal{U}}^{-1}(a,\Lambda)$

$\mathcal{U}(a,\Lambda)\,\hat{P}^{\mu}\,\hat{\mathcal{U}}^{-1}(a,\Lambda)$

2) take (a,Λ) infinitesimal & compute algebra

STEP 1

$$\hat{\mathcal{U}}(a,\Lambda)\,\hat{\mathcal{U}}(\epsilon,\mathbb{1}+\hat{\omega})\,\hat{\mathcal{U}}^{-1}(a,\Lambda)$$

$$\|$$

$$\hat{\mathcal{U}}(a,\Lambda)\,\hat{\mathcal{U}}(\epsilon,\mathbb{1}+\hat{\omega})\,\hat{\mathcal{U}}(-\Lambda^{-1}a,\Lambda^{-1})$$

$$\|$$

$$\hat{\mathcal{U}}(a,\Lambda)\,\hat{\mathcal{U}}\left(-(1+\hat{\omega})\Lambda^{-1}a+\epsilon,(1+\hat{\omega})\Lambda^{-1}\right)$$

$$\|$$

$$\hat{\mathcal{U}}\left(-\Lambda(1+\hat{\omega})\Lambda^{-1}a+\Lambda\epsilon+a,\Lambda(1+\hat{\omega})\Lambda^{-1}\right)$$

$$\|$$

$$\hat{\mathcal{U}}\left(\cancel{-a}-\Lambda\hat{\omega}\Lambda^{-1}a+\Lambda\epsilon\cancel{+a},1+\Lambda\hat{\omega}\Lambda^{-1}\right)$$

$$\|$$

$$\| $$

$$\simeq 1 + \frac{i}{2}\left(1 + \Lambda\hat{\omega}\Lambda^{-1}\right)_{\mu\nu}\hat{J}^{\mu\nu}$$

$$+ i\left(\Lambda\epsilon - \Lambda\hat{\omega}\Lambda^{-1}a\right)_{\mu}\hat{P}^{\mu}$$

On the other hand

$$\hat{U}(a,\Lambda)\,\hat{U}(\epsilon, 1+\hat{\omega})\,\hat{U}^{-1}(a,\Lambda)$$

$$\|$$

$$\hat{U}(a,\Lambda)\left(1 + \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu} + i\epsilon_{\mu}\hat{P}^{\mu}\right)\hat{U}^{-1}(a,\Lambda)$$

$$\wr\wr$$

$$1 + \frac{i}{2}\omega_{\mu\nu}\,\hat{U}(a,\Lambda)\hat{J}^{\mu\nu}\hat{U}^{-1}(a,\Lambda)$$

$$+ i\epsilon_{\mu}\,\hat{U}(a,\Lambda)\hat{P}^{\mu}\hat{U}^{-1}(a,\Lambda)$$

$\Rightarrow$ need to compare the $\omega_{\mu\nu}$ & ϵ_{μ} terms

Opening up first expression found:

$$1+\frac{i}{2}\Big(\cancel{g_{\mu\nu}}+\Lambda_\alpha{}^\mu\,\omega_{\mu\nu}(\Lambda^{-1})^\nu{}_\beta\Big)\hat{J}^{\alpha\beta}+i\,\Lambda_\alpha{}^\mu\epsilon_\mu\,\hat{P}^\alpha$$

symmetric

$$-i\,\Lambda_\alpha{}^\mu\,\omega_{\mu\nu}(\Lambda^{-1})^\nu{}_\rho\,a^\rho\,\hat{P}^\alpha$$

$$\simeq 1+\frac{i}{2}\,\omega_{\mu\nu}\Big[\Lambda_\alpha{}^\mu(\Lambda^{-1})^\nu{}_\beta\,\hat{J}^{\alpha\beta}-2\Lambda_\alpha{}^\mu(\Lambda^{-1})^\nu{}_\rho\,a^\rho\,\hat{P}^\alpha\Big]$$

$$+\,i\epsilon_\mu\left(\Lambda_\alpha{}^\mu\,\hat{P}^\alpha\right)$$

Remembering that from

$$\Lambda^T g\Lambda=g\quad\Rightarrow(\Lambda^{-1})^\mu{}_\nu=\Lambda_\nu{}^\mu$$

$$\simeq 1+\frac{i}{2}\,\omega_{\mu\nu}\Big[\Lambda_\alpha{}^\mu\Lambda_\beta{}^\nu\,\hat{J}^{\alpha\beta}-2\underbrace{\Lambda_\alpha{}^\mu\Lambda_\beta{}^\nu\,a^\beta\,\hat{P}^\alpha}_{\text{use antisymm of }\omega_{\mu\nu}}\Big]$$

$$+\,i\epsilon_\mu\left(\Lambda_\alpha{}^\mu\,\hat{P}^\alpha\right)$$

$$\overset{!}{\simeq} 1 + \frac{i}{2}\omega_{\mu\nu}\underbrace{\left[\Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu\left(\hat{J}^{\alpha\beta} - a^\beta \hat{P}^\alpha + a^\alpha \hat{P}^\beta\right)\right]}_{\text{must be } \hat{U}(a,\Lambda)\hat{J}^{\mu\nu}\hat{U}^{-1}(a,\Lambda)}$$

$$+ i\epsilon_\mu \underbrace{\left(\Lambda_\alpha{}^\mu \hat{P}^\alpha\right)}_{\text{must be } \hat{U}(a,\Lambda)\hat{P}^\mu\hat{U}^{-1}(a,\Lambda)}$$

STEP 2

Take now $a \ll 1$, $\Lambda \simeq 1 + \hat{\omega}'$

We need $\hat{U}^{-1}(a, 1+\hat{\omega}') = \hat{U}\left(-(1+\omega')^{-1}a, (1+\omega')^{-1}\right)$

$$\overset{!}{=} \hat{U}\left(-(1-\omega')a, (1-\omega')\right)$$

$$\overset{!}{=} \hat{U}(-a, 1-\omega')$$

$$\overset{!}{\simeq} 1 - \frac{i}{2}\omega'_{\mu\nu}\hat{J}^{\mu\nu} - i a_\mu \hat{P}^\mu$$

Then

$$\hat{U}(a,\Lambda)\hat{J}^{\mu\nu}\hat{U}^{-1}(a,\Lambda)$$

$$\| $$

$$\left(1+\frac{i}{2}(\omega'\cdot\hat{J})+i(a\cdot\hat{P})\right)\hat{J}^{\mu\nu}\left(1-\frac{i}{2}(\omega'\cdot\hat{J})-i(a\cdot\hat{P})\right)$$

$$\| $$

$$\hat{J}^{\mu\nu}+\frac{i}{2}\left[(\omega'\cdot\hat{J})\hat{J}^{\mu\nu}-\hat{J}^{\mu\nu}(\omega'\cdot\hat{J})\right]+i\left[(a\cdot\hat{P})\hat{J}^{\mu\nu}-\hat{J}^{\mu\nu}(a\cdot\hat{P})\right]$$

$$\| $$

$$\hat{J}^{\mu\nu}+\frac{i}{2}\omega'_{\alpha\beta}\left[\hat{J}^{\alpha\beta},\hat{J}^{\mu\nu}\right]+i\,a_\beta\left[\hat{P}^\beta,\hat{J}^{\mu\nu}\right]$$

$\|$ according to previous computation (pg. 32)

$$\Lambda_\alpha{}^\mu\Lambda_\beta{}^\nu\left(\hat{J}^{\alpha\beta}+a^\alpha\hat{P}^\beta-a^\beta\hat{P}^\alpha\right)$$

$$\| $$

$$\left(\delta_\alpha{}^\mu+\omega'_\alpha{}^\mu\right)\left(\delta_\beta{}^\nu+\omega'_\beta{}^\nu\right)\left(\hat{J}^{\alpha\beta}+a^\alpha\hat{P}^\beta-a^\beta\hat{P}^\alpha\right)$$

$$\| $$

$$\|$$

$$\hat{J}^{\mu\nu} + a^{\mu}\hat{P}^{\nu} - a^{\nu}\hat{P}^{\mu} + \omega'_{\alpha}{}^{\mu}\hat{J}^{\alpha\nu} + \omega'_{\beta}{}^{\nu}\hat{J}^{\mu\beta}$$

$$\|$$

$$\hat{J}^{\mu\nu} + \omega'_{\alpha\beta}\left[\hat{J}^{\alpha\nu}g^{\beta\mu} + \hat{J}^{\mu\beta}g^{\alpha\nu}\right] + a_{\alpha}\left(g^{\alpha\mu}\hat{P}^{\nu} - g^{\alpha\nu}\hat{P}^{\mu}\right)$$

$\|$ use antisymmetry of $\omega'_{\alpha\beta}$

$$\hat{J}^{\mu\nu} + \frac{\omega'_{\alpha\beta}}{2}\left[\hat{J}^{\alpha\nu}g^{\beta\mu} + \hat{J}^{\mu\beta}g^{\alpha\nu} - \hat{J}^{\beta\nu}g^{\alpha\mu} - \hat{J}^{\mu\alpha}g^{\beta\nu}\right] + a_{\alpha}\left[g^{\alpha\mu}\hat{P}^{\nu} - g^{\alpha\nu}\hat{P}^{\mu}\right]$$

Comparing the two results:

$$\begin{cases} i\left[\hat{J}^{\alpha\beta}, \hat{J}^{\mu\nu}\right] = g^{\beta\mu}\hat{J}^{\alpha\nu} + g^{\alpha\nu}\hat{J}^{\mu\beta} - g^{\alpha\mu}\hat{J}^{\beta\nu} \\ \qquad\qquad\qquad\qquad - g^{\beta\nu}\hat{J}^{\mu\alpha} \\ \\ i\left[\hat{P}^{\alpha}, \hat{J}^{\mu\nu}\right] = g^{\alpha\mu}\hat{P}^{\nu} - g^{\alpha\nu}\hat{P}^{\mu} \end{cases}$$

We can now repeat with $\hat{\mathcal{U}}(a, 1+\omega')\,\hat{P}^{\alpha}\,\hat{\mathcal{U}}^{-1}(a, 1+\omega')$

$$\hat{\mathcal{U}}(a, 1+\omega')\,\hat{P}^{\alpha}\,\hat{\mathcal{U}}^{-1}(a, 1+\omega') = \Lambda_{\mu}{}^{\alpha}\,\hat{P}^{\mu}$$

$$\downarrow$$

$$\left[1 + \frac{i}{2}(\omega'\hat{J}) + i(a\cdot\hat{P})\right]\hat{P}^{\alpha}\left[1 - \frac{i}{2}(\omega\cdot\hat{J}) - i(a\cdot\hat{P})\right] = \hat{P}^{\alpha} + \omega'_{\mu}{}^{\alpha}\hat{P}^{\mu}$$

$$\downarrow$$

$$\hat{P}^{\alpha} + \frac{i}{2}\omega'_{\mu\nu}\left[\hat{J}^{\mu\nu}, \hat{P}^{\alpha}\right] + i\,a_{\mu}\left[\hat{P}^{\mu}, \hat{P}^{\alpha}\right] = \hat{P}^{\alpha} + \omega'_{\mu\nu}g^{\nu\alpha}\hat{P}^{\mu}$$

$$\Big|$$

$$\downarrow$$

$$\hat{P}^\alpha + \frac{i}{2}\omega'_{\mu\nu}\left[\hat{J}^{\mu\nu}, \hat{P}^\alpha\right] + i\, a_\mu \left[\hat{P}^\mu, \hat{P}^\alpha\right] = \hat{P}^\alpha + \frac{\omega'_{\mu\nu}}{2}\left(g^{\nu\alpha}\hat{P}^\mu - g^{\mu\alpha}\hat{P}^\nu\right)$$

$$\Downarrow$$

$$\begin{cases} i\left[\hat{J}^{\mu\nu}, \hat{P}^\alpha\right] = g^{\nu\alpha}\hat{P}^\mu - g^{\mu\alpha}\hat{P}^\nu & \text{(as before)} \\ \left[\hat{P}^\mu, \hat{P}^\alpha\right] = 0 \end{cases}$$

$$\Downarrow$$

fundamental result:
the components of the 4-momentum operator form a compatible set of operators!

WIGNER'S CLASSIFICATION

Idea:

PARTICLE $\leftrightarrow$ IRREDUCIBLE REPRESENTATION
POINCARÉ GROUP

How do we achieve the classification?

Inspiration from $SO(3)$ & $SU(2)$: we identify as many <u>Casimir generators</u> as possible

$\downarrow$

operators that commute with all the elements of the algebra

Poincaré algebra → pgs. 35 & 36

CLAIM → $\boxed{\text{first Casimir is } \hat{P}^2 \equiv \hat{P}^\mu \hat{P}_\mu}$

Proof:

a) Call $C^{\alpha\beta\mu} \equiv [\hat{J}^{\alpha\beta}, \hat{P}^\mu] = \hat{J}^{\alpha\beta}\hat{P}^\mu - \hat{P}^\mu \hat{J}^{\alpha\beta}$

Then

$$[\hat{J}^{\alpha\beta}, \hat{P}^2] = \underbrace{\hat{J}^{\alpha\beta}\hat{P}^\mu P_\mu}_{C^{\alpha\beta\mu} + \hat{P}^\mu \hat{J}^{\alpha\beta}} - P_\mu \underbrace{\hat{P}^\mu \hat{J}^{\alpha\beta}}_{\hat{J}^{\alpha\beta}\hat{P}^\mu - C^{\alpha\beta\mu}}$$

$$= C^{\alpha\beta\mu}\hat{P}_\mu + \cancel{\hat{P}^\mu \hat{J}^{\alpha\beta}\hat{P}_\mu} - \cancel{\hat{P}^\mu \hat{J}^{\alpha\beta}\hat{P}_\mu} + \hat{P}_\mu C^{\alpha\beta\mu}$$

$$= -i\left(g^{\mu\beta}\hat{P}^\alpha - g^{\mu\alpha}\hat{P}^\beta\right)\hat{P}_\mu - i\,\hat{P}_\mu\left(g^{\mu\beta}\hat{P}^\alpha - g^{\mu\alpha}\hat{P}^\beta\right)$$

$$= -i\left[\cancel{\hat{P}^\alpha\hat{P}^\beta} - \cancel{\hat{P}^\beta\hat{P}^\alpha} + \cancel{\hat{P}^\beta\hat{P}^\alpha} - \cancel{\hat{P}^\alpha\hat{P}^\beta}\right]$$

$$= 0$$

b) $$[\hat{P}^\alpha, \hat{P}^2] = \underbrace{\hat{P}^\alpha\hat{P}^\mu}_{\text{they commute}}\hat{P}_\mu - \hat{P}_\mu\hat{P}^\mu\hat{P}^\alpha$$

$$= \hat{P}^\mu\hat{P}^\alpha\hat{P}_\mu - \hat{P}_\mu\hat{P}^\mu\hat{P}^\alpha$$

$$= \hat{P}^2\hat{P}^\alpha - \hat{P}^2\hat{P}^\alpha = 0$$

Now, from $[\hat{P}^\alpha, \hat{P}^\beta] = 0 \rightarrow$ Complete set of observables

$$\hat{P}^\mu|p,\sigma\rangle = p^\mu|p,\sigma\rangle$$

any other quantum n° needed to completely describe the state.

Action of 1st Casimir:

$$\hat{P}^2 |p,\sigma\rangle = p^2 |p,\sigma\rangle = m^2 |p,\sigma\rangle$$

we use the mass to classify a particle

Lorentz transformation on states

What happens applying $\hat{U}(0,\Lambda) \equiv \hat{U}(\Lambda)$ on $|p,\sigma\rangle$?

$$\hat{P}^\mu \hat{U}(\Lambda)|p,\sigma\rangle = \hat{U}(\Lambda)\left[\hat{U}^{-1}(\Lambda)\hat{P}^\mu \hat{U}(\Lambda)\right]|p,\sigma\rangle$$

from pg. 32

$$\hat{U}(\Lambda)\hat{P}^\mu \hat{U}^{-1}(\Lambda) = \Lambda_\alpha{}^\mu \hat{P}^\alpha$$

$$\Lambda_\alpha{}^\mu = (\Lambda^{-1})^\mu{}_\alpha$$

$$\Downarrow$$

$$\hat{U}^{-1}(\Lambda)\hat{P}^\mu \hat{U}(\Lambda) = \hat{U}(\Lambda^{-1})\hat{P}^\mu \hat{U}^{-1}(\Lambda^{-1}) = \Lambda^\mu{}_\alpha \hat{P}^\alpha$$

Then

$$\hat{P}^\mu \hat{U}(\Lambda)|p,\sigma\rangle = \Lambda^\mu{}_\alpha \hat{U}(\Lambda)\hat{P}^\alpha|p,\sigma\rangle$$
$$= \Lambda^\mu{}_\alpha p^\alpha \hat{U}(\Lambda)|p,\sigma\rangle$$
$$= (\Lambda p)^\mu \hat{U}(\Lambda)|p,\sigma\rangle$$

$\Longrightarrow$ $\hat{U}(\Lambda)|p,\sigma\rangle$ has momentum Λp

$\Downarrow$

we write it as a general combination of kets $|\Lambda p,\sigma'\rangle$ where σ' allowed to be different from σ :

$$\boxed{\hat{U}(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma\sigma'}(\Lambda,p)\,|\Lambda p,\sigma'\rangle}$$

We still do not know what σ is

We now use a trick due to Wigner:

there are frames in which p^μ is particularly simple:

a. <u>massive particles</u> $p_0 = \begin{bmatrix} m \\ \vec{0} \end{bmatrix}$

(rest frame)

b. <u>massless particles</u> $p_0 = \begin{bmatrix} E \\ 0 \\ 0 \\ E \end{bmatrix}$

Any p^μ can be obtained from p_0 via a Lorentz transformation:

$$p = L_p p_0$$

Now, DEFINE quantum n° σ as be left invariant by L_p,

$$|p,\sigma\rangle = N_p\, \hat{U}(L_p)\, |p_0,\sigma\rangle$$

normalization

To classify σ (= understand what it is) we use another trick due to Wigner: since σ is defined in the frame in which momentum $= p_0$, we look at the Lorentz transformations that leave us in this frame but modify σ.

Leaving frame the same: $M p_0 = p_0$

element of LITTLE GROUP of p_0

By definition, the only effect of M is to modify σ, that thus transform in an irreducible representation of the Little group:

$$\hat{U}(M)\,|p_0,\sigma\rangle = \sum_{\sigma'} D_{\sigma\sigma'}(M)\,|p_0,\sigma'\rangle$$

representation little group

$\Downarrow$

we need to study what the little group is to understand what σ is

How do we connect this with what we saw before?

Fundamental observation:

$$\Lambda p = L_{\Lambda p}\, p_0$$

$$\Downarrow$$

$$\Lambda L_p\, p_0 = L_{\Lambda p}\, p_0$$

$$\Downarrow$$

$$\underbrace{\left(L_{\Lambda p}^{-1}\, \Lambda\, L_p\right)}_{M}\, p_0 = p_0$$

M = element little group!

But then

$$
\begin{aligned}
\hat{U}(\Lambda)|p,\sigma\rangle &= N_p\,\hat{U}(\Lambda)\,\hat{U}(L_p)|p_0,\sigma\rangle \\
&= N_p\,\hat{U}(\Lambda L_p)|p_0,\sigma\rangle \\
&= N_p\,\hat{U}(L_{\Lambda p}M)|p_0,\sigma\rangle \\
&= N_p\,\hat{U}(L_{\Lambda p})\,\hat{U}(M)|p_0,\sigma\rangle \\
&= N_p\,\hat{U}(L_{\Lambda p})\sum_{\sigma'}\mathcal{D}_{\sigma\sigma'}(M)\,|p_0,\sigma'\rangle \\
&= N_p\sum_{\sigma'}\mathcal{D}_{\sigma\sigma'}(M)\,\underbrace{\hat{U}(L_{\Lambda p})|p_0,\sigma'\rangle}_{\frac{|\Lambda p,\sigma'\rangle}{N_{\Lambda p}}} \\
&= \frac{N_p}{N_{\Lambda p}}\sum_{\sigma'}\mathcal{D}_{\sigma\sigma'}(M)\,|\Lambda p,\sigma'\rangle \\
&= \sum_{\sigma'}C_{\sigma\sigma'}(\Lambda,p)\,|\Lambda p,\sigma'\rangle
\end{aligned}
$$

pg. 42

This means that the irreducible representations of the Poincaré group ($C_{\sigma\sigma'}$) are completely determined in terms of representations of the little group ($D_{\sigma\sigma'}$)

Looking at the little group we also find the second Casimir:

$$M p_0 = p_0 \implies \left(\delta^\mu{}_\nu + \hat{\omega}^\mu{}_\nu\right) p_0^\nu = p_0^\mu$$

$$\Downarrow$$

$$\hat{\omega}_{\mu\nu}\, p_0^\nu = 0$$

$$\Downarrow$$

$$\frac{1}{2}\left(\omega_{\alpha\beta} M^{\alpha\beta}\right)_{\mu\nu} p_0^\nu = 0$$

with $(M^{\alpha\beta})_{\mu\nu} = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu$

$$\downarrow$$

$$\omega_{\alpha\beta}\left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu\right) p_0^\nu = 0$$

$$\downarrow$$

$$\omega_{\mu\nu}\, p_0^\nu = 0$$

parameters

solution

arbitrary

$$\omega_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}\, p_0^\alpha h^\beta$$

At quantum level

$$\hat{\mathcal{U}}(M) = e^{\frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}} = e^{\frac{i}{2}\epsilon_{\mu\nu\alpha\beta}p_0^\alpha h^\beta \hat{J}^{\mu\nu}}$$

$$= e^{ih^\beta \hat{W}_\beta}$$

where $\hat{W}_\beta \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\, p_0^\alpha\, \hat{J}^{\mu\nu}$

PAULI-LUBANSKI vector

Important results :

1. $[\hat{W}^\alpha, \hat{P}^\beta] = 0$

2. $\hat{W}^2$ is the SECOND CASIMIR of the Poincaré group

MASSIVE PARTICLES

$m > 0 \rightarrow$ can choose $p_0 = \begin{bmatrix} m \\ \vec{0} \end{bmatrix}$ (rest frame)

Little Group $\rightarrow$ $M p_0 = p_0$

$$\Updownarrow$$

$$M = \left[\begin{array}{c|c} 0 & \vec{0} \\ \hline \vec{0} & \text{rotation} \end{array}\right]$$

$\Rightarrow$ Little group $= SO(3)$

Pauli-Lubanski vector:

$$W_\alpha = \frac{1}{2} \epsilon_{\mu\nu\beta\alpha} p_0^\beta \hat{J}^{\mu\nu} = \frac{m}{2} \underbrace{\epsilon_{\mu\nu\beta\alpha} \delta^{\beta 0} \hat{J}^{\mu\nu}}_{\epsilon_{\mu\nu 0 \alpha}}$$

$$\Downarrow$$

μ, ν, α must be spatial indices!

$$\Rightarrow W_k = \frac{m}{2} \underbrace{\epsilon_{ij0k} \hat{J}^{ij}}$$

purely spatial $\Rightarrow$ must select angular momentum operators

$\Downarrow$

in rest frame we obtain

Spin ops.

$= -m S_k$

Casimir $\rightarrow$ $\hat{\vec{W}}^2 = -m^2 \hat{\vec{S}}^2$

$\Rightarrow$ we need mass $\oplus$ spin to classify particles (as in NRQM!)

MASSLESS PARTICLES

$m=0 \Rightarrow$ no rest frame, but we can pick up

$$p_0^\mu = \begin{bmatrix} E \\ 0 \\ 0 \\ E \end{bmatrix}$$

Little group: write $M^\mu{}_\nu = \delta^\mu{}_\nu + i\alpha^\mu{}_\nu$

Imposing $Mp_0 = p_0$ we get

$$\alpha^\mu{}_\nu = \omega_1 (-i) \begin{bmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{bmatrix} \hat{A}$$

$$+\omega_2 (-i) \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \hat{B}$$

$$+\omega_3 (-i) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \hat{J}_3$$

$\Rightarrow$ we obtain $\hat{J}_3 \oplus$ two more generators.

This group is called "Euclidian group in 2-dim"

$$[\hat{A},\hat{B}]=0 \;;\; [\hat{J}_3,\hat{A}]=i\hat{B}$$
$$[\hat{J}_3,\hat{B}]=-i\hat{A}$$

$\hat{A}$ & $\hat{B}$ compatible and rotations around $\hat{e}_z$ transform one into the other.

What are the quantum numbers of a massless particle under $\hat{A}$ & $\hat{B}$?

Take $|a,b\rangle$ such that $\begin{cases} \hat{A}|a,b\rangle = a|a,b\rangle \\ \hat{B}|a,b\rangle = b|a,b\rangle \end{cases}$

Now, under a $\hat{J}_3$ rotation, we have

$$\hat{U}(\theta)\,\hat{A}\,\hat{U}^{-1}(\theta) = \left(1 + i\theta\hat{J}_3 - \frac{\theta^2}{2}\hat{J}_3^2\right)\hat{A}\left(1 - i\theta\hat{J}_3 - \frac{\theta^2}{2}\hat{J}_3^2\right)$$

$$= \hat{A} + i\theta\left[\hat{J}_3, \hat{A}\right]$$

$$+ \theta^2\left[\hat{J}_3\hat{A}\hat{J}_3 - \frac{1}{2}\hat{J}_3^2\hat{A} - \frac{1}{2}\hat{A}\hat{J}_3^2\right]$$

use commut. relations

$$= \hat{A} - \theta\hat{B}$$

$$+ \frac{\theta^2}{2}\Big[\cancel{2\hat{J}_3\hat{A}\hat{J}_3} - i\hat{J}_3\hat{B} - \cancel{\hat{J}_3\hat{A}\hat{J}_3}$$

$$- \cancel{\hat{J}_3\hat{A}\hat{J}_3} + i\hat{B}\hat{J}_3\Big]$$

$$= \hat{A} - \theta\hat{B} - \frac{i\theta^2}{2}\left[\hat{J}_3, \hat{B}\right]$$

$$= \hat{A} - \theta\hat{B} - \frac{\theta^2}{2}\hat{A} + \dots$$

$$\simeq \cos\theta\,\hat{A} - \sin\theta\,\hat{B}$$

Also

$$\hat{U}(\theta)\hat{B}\hat{U}^{-1}(\theta) = \hat{A}\sin\theta + \hat{B}\cos\theta$$

Thus, for arbitrary θ, we can define

$$|a,b\rangle_\theta \equiv \hat{U}^{-1}(\theta)\,|a,b\rangle$$

$$\Longrightarrow \begin{cases} \hat{A}\,|a,b\rangle_\theta = (a\cos\theta - b\sin\theta)\,|a,b\rangle_\theta \\ \hat{B}\,|a,b\rangle_\theta = (a\sin\theta + b\cos\theta)\,|a,b\rangle_\theta \end{cases}$$

Thus, if $\exists$ $a \neq 0$ or $b \neq 0$, we have a continuum of states

$\Rightarrow$ we should observe in nature an additional continuous quantum number that is not observed $\Rightarrow$ $\boxed{\text{need to admit } a = 0 = b}$

When $a=0=b$, only $\hat{J}_3$ is left

$\Rightarrow$ states transform under "little group" $SO(2)$

Since this $SO(2)$ is a subgroup of the $SO(3)$ contained in $SO(1,3)_+$ its eigenvalues are quantized

$$\Downarrow$$

$$\hat{J}_3 \text{ has eigenvalues } \frac{k}{2}, \; k \in \mathbb{Z}$$

Pauli-Lubanski: using the definition

$$\hat{W}_\alpha = \frac{1}{2} \epsilon_{\mu\nu\beta\alpha} p_0^\beta \hat{J}^{\mu\nu}$$

$$= -\epsilon_{12\beta\alpha} p_0^\beta \hat{J}^3$$

$$= \begin{bmatrix} -E\hat{J}_3 \\ 0 \\ 0 \\ E\hat{J}_3 \end{bmatrix}$$

But we can also write $p_0 = \begin{bmatrix} E \\ E\hat{e}_z \end{bmatrix}$

$$\Rightarrow \hat{W}_\alpha = \begin{bmatrix} -\vec{p}_0 \cdot \vec{J} \\ 0 \\ 0 \\ \vec{p}_0 \cdot \vec{J} \end{bmatrix}$$

$\hookrightarrow$ We are finding that the little group selects as useful quantum number the projection of the angular momentum along $\vec{p}_0$

Putting everything together:

$$\underbrace{p_0^2 = 0 \, ; \quad p_0 \hat{W} = 0 \, ; \quad \hat{W}^2 = 0}$$

$$\Downarrow$$

necessarily $\hat{W}^\alpha = h \, p_0^\alpha$

$$\overset{!}{=} \begin{bmatrix} -\vec{p}_0 \cdot \vec{J} \\ 0 \\ 0 \\ -\vec{p}_0 \cdot \vec{J} \end{bmatrix} = \underbrace{-\frac{\vec{p}_0 \cdot \vec{J}}{|\vec{p}_0|}}_{h} \begin{bmatrix} |\vec{p}_0| \\ 0 \\ 0 \\ |\vec{p}_0| \end{bmatrix}$$

HELICITY

Since $[\hat{W}^\alpha, \hat{P}^\beta] = 0 \Rightarrow$ compatible observables

$\Rightarrow$ label the states as $|p_0, h\rangle$

with $\begin{cases} \hat{P}^\mu |p_0, h\rangle = p_0^\mu |p_0, h\rangle \\ \hat{W}^\mu |p_0, h\rangle = h \, p_0^\mu |p_0, h\rangle \end{cases}$

NORMALIZATION OF STATES

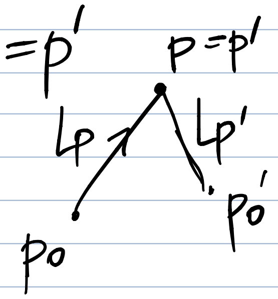
What is the factor N_p appearing in

$$|p,\sigma\rangle = N_p \,\hat{\mathcal{U}}(L_p)\,|p_0,\sigma\rangle \;?$$

Imposе

$$\langle p',\sigma'|\,p,\sigma\rangle = \delta_{\sigma\sigma'}\,\delta^3(\vec{p}-\vec{p}\,')$$

$$= N^*_{p'}\,N_p\,\langle p_0',\sigma'|\,\underbrace{\hat{\mathcal{U}}^\dagger(L_{p'})\,\mathcal{U}(L_p)}_{\text{Choose } p=p'}\,|p_0,\sigma\rangle$$



$$= |N_p|^2\,\delta_{\sigma\sigma'}\,\delta^3(\vec{p}_0-\vec{p}_0\,')$$

Connection between $\delta^3(\vec{p}-\vec{p}')$ & $\delta^3(\vec{p}_0-\vec{p}_0')$?

A Lorentz invariant integration is given by

$$\int d^4p\, \delta(p^2-M^2)\,\theta(p^0) = \int d^3p \int dp^0\, \delta(p^{0\,2}-\vec{p}^{\,2}-M^2)\,\theta(p^0)$$

$$= \int \frac{d^3p}{2p_0}\bigg|_{p_0=\sqrt{\vec{p}^{\,2}+M^2}}$$

Thus if $F(\vec{p})$ is an invariant function, we obtain

$$F(\vec{p}') = \int d^3p\, \delta^3(\vec{p}-\vec{p}')\, F(\vec{p})$$

$$= \underbrace{\int \frac{d^3p}{\sqrt{\vec{p}^{\,2}+M^2}}}_{\text{invariant}} \underbrace{\left(\sqrt{\vec{p}^{\,2}+M^2}\,\delta^3(\vec{p}-\vec{p}')\right)}_{\text{invariant}} \underbrace{F(\vec{p})}_{\text{invariant}}$$

$$\Longrightarrow \quad p^0 \delta^3(\vec{p}-\vec{p}\,') = \text{invariant} = p_0^0 \delta^3(\vec{p}_0-\vec{p}_0\,')$$

Putting all together:

$$\delta_{\sigma\sigma'} \delta^3(\vec{p}-\vec{p}\,') = |N_p|^2 \delta_{\sigma\sigma'} \delta^3(\vec{p}_0-\vec{p}_0\,')$$

$$\overset{!}{=} |N_p|^2 \delta_{\sigma\sigma'} \frac{p^0}{p_0^0} \delta^3(\vec{p}-\vec{p}\,')$$

$$\Longrightarrow \boxed{N_p = \sqrt{\frac{p_0^0}{p^0}}}$$

EXAMPLE : CLASSICAL EM FIELD

We have developed our formalism applied to quantum states, but most of the concepts can be applied in the same way to classical physics.

In particular, it is always true that

$$\Lambda p = \underbrace{\Lambda L_p}_{L_{\Lambda p}} p_0 \Rightarrow \underbrace{(L^{-1}_{\Lambda p} \Lambda L_p)}_{\text{the little group}} p_0 = p_0$$

$$\Rightarrow \quad \Lambda p = L_{\Lambda p} M p_0$$

$$\Downarrow$$

<u>any</u> Lorentz tr. can always be written starting from the little group

In the frame of $p_0 = \begin{bmatrix} E \\ 0 \\ 0 \\ E \end{bmatrix}$ the EM potential can be written as

$$A^{\mu}_{p_0} = \epsilon^{\pm\mu}_{p_0} e^{-ip_0 x} \quad \text{with} \quad \epsilon^{\pm\mu}_{p_0} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{bmatrix}$$

EM field is transverse

The little group transformation is

$$M = \exp \begin{bmatrix} 0 & \omega_1 & \omega_2 & 0 \\ \omega_1 & 0 & -\omega_3 & -\omega_1 \\ \omega_2 & \omega_3 & 0 & -\omega_2 \\ 0 & \omega_1 & \omega_2 & 0 \end{bmatrix}$$

$=$ complicated expression

Applied on $\epsilon^{\pm}_{p_0}$ we get

$$M \epsilon^{\pm\mu}_{p_0} = e^{\pm i\omega_3} \left(\epsilon^{\pm}_{p_0} + f_{\pm}(\omega_1, \omega_2)\, p_0 \right)$$

2 conclusions:

1. the helicity is ± 1

$$\Downarrow$$

physical photon contains both $\epsilon^{0\pm\mu}$

$$\Downarrow$$

contains $h = \pm 1$

2. The part of M associated with ω_1 & ω_2 (A & B generators) cause $\epsilon^{\pm}_{p_0}$ to shift by a quantity proportional to p_0

But then, defining $\epsilon^{\pm}_{p} \equiv L_p \epsilon^{\pm}_{p_0}$, we get

$$\Lambda \epsilon^{\pm}_{p} = \Lambda L_p \epsilon^{\pm}_{p_0} = L_{\Lambda p} M \epsilon^{\pm}_{p_0}$$

$$= L_{\Lambda p} e^{\pm i\omega_3} \left(\epsilon^{\pm}_{p_0} + f_{\pm} p_0 \right)$$

$$= e^{\pm i\omega_3} \left(\epsilon^{\pm}_{\Lambda p} + f_{\pm} \Lambda p \right)$$

This will be fundamental in the quantization of the EM field.

Message: although we write them as 4-vectors, the polarization vectors of the EM field do not transform as 4-vectors under a Lorentz transformation. The shift proportional to the momentum will be the gauge transform.

RELATIVISTIC WAVE EQUATIONS

Now we use Wigner's classification:

For a particle with mass m and spin s we will combine representations of the Lorentz group that contain the chosen spin.

How? Since we seek for wave equations, we will allow

1. for the operator $\hat{P}_\mu = i\partial_\mu = \begin{pmatrix} i\partial_t \\ i\vec{\nabla} \end{pmatrix}$ to appear

2. we'll demand Lorentz covariance (i.e. the wave eqs. look the same in all inertial frames)

SPIN-0 PARTICLE

We have seen that $s=0$ is contained in

$$\begin{cases} (0,0) \leftrightarrow \phi(x) \\ \left(\frac{1}{2},\frac{1}{2}\right) \longleftrightarrow \phi^\mu(x) \end{cases}$$

Only covariants that can be formed are

$$\begin{cases} \hat{p}^\mu \phi_\mu = m\phi \\ \hat{p}^\mu \phi = m\phi^\mu \end{cases}$$

the two constants can always be taken equal by a wave function redefinition

$$\Longrightarrow \hat{p}_\mu(\hat{p}^\mu \phi) = m\,\hat{p}_\mu \phi^\mu \stackrel{\downarrow}{=} m^2\phi$$

$$\Rightarrow (\hat{p}^2 - m^2)\phi = 0$$

KLEIN-GORDON EQ.

Since $\hat{p}^\mu = i\partial^\mu \Rightarrow (\Box + m^2)\phi = 0$

Once $\phi(x)$ is known, we can compute $\phi^\mu(x)$ using $\phi^\mu = \frac{1}{m}\hat{p}^\mu\phi$

$\Rightarrow \phi^\mu$ IS NOT AN INDEPENDENT FIELD!

SOLUTIONS OF KG EQUATION

Since KG = wave equation, plane waves $\|$ complete set of solutions

$$\phi_k = \frac{e^{-ikx}}{\sqrt{2E}}\phi_0 = \frac{e^{-i(k^0 t - \vec{k}\cdot\vec{x})}}{\sqrt{2E}}\phi_0$$

convenient normalization

Inserting in KG $\longrightarrow$ $(k^2-m^2)\,\phi_0 = 0$

$$\Downarrow$$

$k^2 = m^2$ ok! Relativistic dispersion relation!

But this imply

$$k^0 = \pm\sqrt{m^2+\vec{k}^2} = \pm E$$

we have both $k^0>0$ & $k^0<0$ solutions??

Most general solution of KG-equation:

$$\phi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}}\left[a_p^{(+)}\, e^{-i(Et-\vec{p}\cdot\vec{x})} + a_p^{(-)}\, e^{-i(-Et-\vec{p}\cdot\vec{x})}\right]$$

change $\vec{p}\to-\vec{p}$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}}\left[a_p^{(+)}\, e^{-ipx} + a_{-p}^{(-)}\, e^{ipx}\right]$$

Also, computing $\frac{\partial P}{\partial t}$ with probability

$$P = \int d^3x \, |\phi(x)|^2$$

we discover that $\dot{P} \neq 0 \Rightarrow ???$

Conclusion: wave function interpretation of $\phi(x)$ does not make much sense

SPIN-1/2 PARTICLE

We have seen that $S=1/2$ is contained in

$$\left(\tfrac{1}{2},0\right) \longleftrightarrow \xi^{a}$$

$$\left(0,\tfrac{1}{2}\right) \longleftrightarrow \bar{\chi}^{\dot{a}}$$

How can the momentum operator $\hat{p}_\mu$ appear & contract spinorial indices?

We use $\boxed{\hat{p}_{a\dot{a}} \equiv \hat{p}^{\mu}(\sigma_\mu)_{a\dot{a}}}$

relation with $SL(2,\mathbb{C})$

We also raise the indices using ϵ :

$$\hat{p}^{\dot{a}a} = \epsilon^{\dot{a}\dot{b}} \epsilon^{ab} \hat{p}_{b\dot{b}}$$

$$= \hat{p}^{\mu} \underbrace{\left(\epsilon^{\dot{a}\dot{b}} \epsilon^{ab} (\sigma_{\mu})_{b\dot{b}} \right)}_{\equiv (\overline{\sigma_{\mu}})^{\dot{a}a} = (\mathbb{1}, -\vec{\sigma})}$$

Covariant wave eq.

$$\hat{p}_{a\dot{a}} \overline{\chi}^{\dot{a}} = m \xi_a \quad , \quad \hat{p}^{\dot{a}a} \xi_a = m \overline{\chi}^{\dot{a}}$$

Using one in the other

$$\hat{p}_{a\dot{a}} \left(\frac{1}{m} \hat{p}^{\dot{a}b} \xi_b \right) = m \xi_a$$

$$\downarrow$$

$$\hat{p}_{a\dot{a}} \hat{p}^{\dot{a}b} \xi_b = m^2 \xi_a$$

Use now

$$\begin{cases} \hat{P}_{a\dot{a}} = \hat{p}^{\mu}(\sigma_{\mu})_{a\dot{a}} = \begin{pmatrix} \hat{p}^0+\hat{p}^3 & \hat{p}^1-i\hat{p}^2 \\ \hat{p}^1+i\hat{p}^2 & \hat{p}^0-\hat{p}^3 \end{pmatrix}_{a\dot{a}} \\ \hat{p}^{\dot{a}b} = \hat{p}^{\mu}(\overline{\sigma_{\mu}})^{\dot{a}a} = \begin{pmatrix} \hat{p}^0-\hat{p}^3 & -\hat{p}^1+i\hat{p}^2 \\ -\hat{p}^1-i\hat{p}^2 & \hat{p}^0+\hat{p}^3 \end{pmatrix}^{\dot{a}b} \end{cases}$$

$$\Longrightarrow \hat{P}_{a\dot{a}}\,\hat{p}^{\dot{a}b} = \left[(\hat{p}^0)^2-(\hat{p}^3)^2-(\hat{p}^1)^2-(\hat{p}^2)^2\right]\delta_a^{\ b}$$

$$= \hat{p}^2\,\delta_a^{\ b}$$

$$\Longrightarrow \begin{cases} (\hat{p}^2-m^2)\,\xi_a = 0 \\ \oplus \\ (\hat{p}^2-m^2)\,\overline{\chi}^{\dot{a}} = 0 \end{cases}$$

$\mapsto$ both spinors satisfy the KG eq.

For $m \neq 0$ both $\bar{\chi}_{\dot{a}}$ & ξ_a are necessary to write down a covariant equation

$\Longrightarrow$ massive $s = 1/2$ particles contain both LH & RH spinors

Leaving implicit the indices, we can write

$$\begin{cases} i\,\sigma_\mu \partial^\mu \bar{\chi} = m\,\xi \\ i\,\bar{\sigma}_\mu \partial^\mu \xi = m\,\bar{\chi} \end{cases}$$

DIRAC EQUATION
(in Weyl form)

Equivalent and more compact way to present the same physics:

$$\psi \equiv \begin{bmatrix} \xi \\ \bar{\chi} \end{bmatrix}$$

Dirac spinor
(or 4-spinor)

by construction in the $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$

In terms of ψ

$$\underbrace{\left[\begin{array}{c|c} -m & i\sigma^\mu\partial_\mu \\ \hline i\bar{\sigma}^\mu\partial_\mu & -m \end{array}\right]}\psi = 0$$

$$\left\{ i\partial_\mu \underbrace{\left[\begin{array}{c|c} 0 & \sigma^\mu \\ \hline \bar{\sigma}^\mu & 0 \end{array}\right]}_{\gamma^\mu} - m \right\}\psi = 0$$

, Feynman slash notation:

$$a_\mu \gamma^\mu = \not{a}$$

$\Downarrow$

$$\boxed{(i\not{\partial} - m)\psi = 0}$$

DIRAC EQUATION

$\Downarrow$

1st order equation

$\swarrow$

Can define a conserved prob. density

$$\rho = |\psi|^2$$

PROPERTIES OF THE γ MATRICES

1. $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ $\leftarrow$ CLIFFORD ALGEBRA

2. $\mu = 0 = \nu \Rightarrow (\gamma^0)^2 = \mathbb{1}$

3. $\mu = i = \nu \Rightarrow (\gamma^i)^2 = -\mathbb{1}$

4. Hamiltonian form Dirac eq.

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$(i\gamma^0\partial_t + i\vec{\gamma}\cdot\vec{\nabla} - m)\psi = 0$$

$$i\gamma^0\partial_t\psi = (-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi$$

$$i\partial_t\psi = \gamma^0(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi$$

$$= (\underbrace{-i\gamma^0\vec{\gamma}\cdot\vec{\nabla}}_{\vec{\alpha}\cdot\vec{p}} + \underbrace{m\gamma^0}_{\beta})\psi$$

(historical notation)

$$= \hat{H}\psi$$

5. Using $\gamma^{0\dagger} = \gamma^0 \Rightarrow \beta^\dagger = \beta$
$\gamma^0 \vec{\gamma}^\dagger \gamma^0 = \vec{\gamma} \Rightarrow \vec{\alpha}^\dagger = \vec{\alpha}$ } from direct computation

$\Rightarrow \hat{H}^\dagger = \hat{H}$ as we want

OBS: all matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ $\oplus$ $\gamma^{0\dagger} = \gamma^0$ $\oplus$ $\gamma^0 \vec{\gamma}^\dagger \gamma^0 = \vec{\gamma}$ are called "γ-matrices". The expression $\gamma^\mu = \left[\begin{array}{c|c} 0 & \sigma^\mu \\ \hline \bar{\sigma}^\mu & 0 \end{array}\right]$ is just one of the possibilities.

SOLUTIONS OF DIRAC EQUATION

Repeating what has been done for the KG eq. (pg 66)

INTRODUCE PLANE WAVES OF POSITIVE & NEGATIVE ENERGY

$$\underbrace{\psi_p^+ = \frac{1}{\sqrt{2E}} u_p e^{-ipx} \quad ; \quad \psi_{-p}^- = \frac{1}{\sqrt{2E}} v_p e^{ipx}}$$

Most general solution:

$$\psi = \int \frac{d^3p}{(2\pi)^3} \left(\psi_p^+ + \psi_{-p}^- \right)$$

Applying $(i\partial\!\!\!/ - m)$ to ψ_p^+ & ψ_{-p}^- :

$$(p\!\!\!/ - m) u_p = 0 \quad ; \quad (p\!\!\!/ + m) v_p = 0$$

Solutions?

Write $u_p \equiv \begin{pmatrix} \alpha \\ \bar{\alpha} \end{pmatrix}$; $v_p = \begin{pmatrix} \beta \\ \bar{\beta} \end{pmatrix}$

$$\Downarrow$$

$$(E - \vec{\sigma}\cdot\vec{p})\bar{\alpha} = m\alpha \qquad (E - \vec{\sigma}\cdot\vec{p})\bar{\beta} = -m\beta$$
$$(E + \vec{\sigma}\cdot\vec{p})\alpha = m\bar{\alpha} \qquad (E + \vec{\sigma}\cdot\vec{p})\beta = -m\bar{\beta}$$

Non-relativistic limit : $E \simeq m$

$$\Rightarrow \quad \bar{\alpha} = \frac{(E + \vec{\sigma}\cdot\vec{p})\alpha}{m} \simeq \frac{m}{m}\alpha + \mathcal{O}(p)$$

$$\simeq \alpha$$

$$\bar{\beta} = \frac{(E + \vec{\sigma}\cdot\vec{p})\beta}{-m} \simeq -\frac{m}{m}\beta + \mathcal{O}(p)$$

$$\simeq -\beta$$

$\Rightarrow$ only 1 Weyl spinor needed to describe a NR spinor, as was done in NRQM

Focus on positive energy sector

(for the negative energy we just need to do $m \to -m$)

To make the NR limit easy define

$$\underbrace{\phi \equiv \alpha + \bar{\alpha}}_{\text{"large component"}} \; ; \; \underbrace{\chi \equiv \bar{\alpha} - \alpha}_{\text{"small component"}}$$

(tends to vanish in NR limit)

$\to$ we obtain

$$\left.\begin{aligned} (E-m)\phi - \vec{\sigma}\cdot\vec{p}\,\chi &= 0 \\ (E+m)\chi - \vec{\sigma}\cdot\vec{p}\,\phi &= 0 \end{aligned}\right\} \underbrace{\left(\begin{array}{c|c} m & \vec{\sigma}\cdot\vec{p} \\ \hline \vec{\sigma}\cdot\vec{p} & -m \end{array}\right)}_{\hat{H} \text{ in this basis}} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = E \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

Remembering $\hat{H} = \gamma^0 \vec{\gamma}\cdot\vec{p} + \gamma^0 m$

$$\Downarrow$$

$$\gamma^0 = \left[\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right] ; \quad \vec{\gamma} = \left[\begin{array}{c|c} 0 & \vec{\sigma} \\ \hline -\vec{\sigma} & 0 \end{array}\right]$$

in this basis

Finding ϕ & χ:

1. write $\vec{p} = |\vec{p}| \left(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\right)$

$$\Rightarrow \quad \vec{\sigma}\cdot\vec{p} = |\vec{p}| \begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{bmatrix}$$

2. Eigenstates of $\vec{\sigma}\cdot\vec{p}$:

$$|\vec{p}| \leftrightarrow \xi_+ = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\varphi} \end{bmatrix} ; \quad -|\vec{p}| \leftrightarrow \xi_- = \begin{bmatrix} -\sin\frac{\theta}{2}\, e^{-i\varphi} \\ \cos\frac{\theta}{2} \end{bmatrix}$$

3. Seek for solutions

$$\begin{pmatrix} \phi_\pm \\ \chi_\pm \end{pmatrix} \equiv \begin{pmatrix} \alpha_\pm \xi_\pm \\ \beta_\pm \xi_\pm \end{pmatrix}$$

$\downarrow$

acting on this, the $\hat{H}$ operator is

$$\left(\begin{array}{c|c} m & \pm|\vec{p}| \\ \hline \pm|\vec{p}| & -m \end{array}\right) \begin{pmatrix} \alpha_\pm \xi_\pm \\ \beta_\pm \xi_\pm \end{pmatrix} = E \begin{pmatrix} \alpha_\pm \xi_\pm \\ \beta_\pm \xi_\pm \end{pmatrix}$$

4. Solutions are

$$\begin{pmatrix} \phi_+ \\ \chi_+ \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{E+m}{2E}}\, \xi_+ \\ \sqrt{\frac{E-m}{2E}}\, \xi_+ \end{pmatrix} ; \quad \begin{pmatrix} \phi_- \\ \chi_- \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{E+m}{2E}}\, \xi_- \\ -\sqrt{\frac{E-m}{2E}}\, \xi_- \end{pmatrix}$$

NON-RELATIVISTIC LIMIT OF DIRAC EQ.

Since we want to obtain a systematic $\frac{1}{c}$ expansion, we reinstate $\hbar$ & c

$\begin{pmatrix} \phi \\ \chi \end{pmatrix}$ basis $\rightarrow$ Convenient for NR limit

Dirac eq. $\rightarrow$ $i\hbar \frac{\partial \psi}{\partial t} = \left[c\gamma^0 \vec{\gamma} \cdot \vec{p} + \gamma^0 mc^2 \right] \psi$

where $\begin{cases} \psi = \begin{bmatrix} \phi \\ \chi \end{bmatrix} \\ \gamma^0 = \left[\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right] \\ \vec{\gamma} = \left[\begin{array}{c|c} 0 & \vec{\sigma} \\ \hline -\vec{\sigma} & 0 \end{array}\right] \end{cases}$

NR limit $\rightarrow$ $E \simeq mc^2 + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3c^2} + \dots$

$\Rightarrow mc^2 =$ dominant contribution to E

$\Downarrow$

factor it out in time evolution
(in NR energy computed on top of mc^2)

$\Downarrow$

$$\boxed{\psi = \psi' e^{-imc^2 t/\hbar}}$$

In terms of ψ' we get:

$$i\hbar \frac{\partial \psi'}{\partial t} = \left[c\gamma^0 \vec{\gamma}\cdot\hat{\vec{p}} + \gamma^0 mc^2 - mc^2 \right] \psi'$$

and we call $\psi' = \begin{bmatrix} \phi' \\ \chi' \end{bmatrix}$

$$\Rightarrow \begin{cases} i\hbar \dfrac{\partial \phi'}{\partial t} = c\,\vec{\sigma}\cdot\hat{\vec{p}}\,\chi' \\ \left(\dfrac{i\hbar}{2mc^2}\dfrac{\partial}{\partial t} + 1\right)\chi' = \dfrac{\vec{\sigma}\cdot\hat{\vec{p}}\,\phi'}{2mc} \end{cases}$$

We can now expand order by order in $\frac{1}{c}$:

1. <u>Keep terms up to</u> $\mathcal{O}\left(\frac{1}{c}\right)$

$$\chi' \simeq \frac{\vec{\sigma}\cdot\hat{\vec{p}}\,\phi'}{2mc}$$

$$\Downarrow$$

$$\underbrace{i\hbar \frac{\partial \phi'}{\partial t} = \frac{\cancel{c}(\vec{\sigma}\cdot\hat{\vec{p}})^2\phi'}{2m\cancel{c}} = \frac{\hat{\vec{p}}^{\,2}}{2m}\phi'}$$

$\downarrow$

we obtain the right Schrödinger eq. for ϕ'!

Probability density:

$$\rho = |\phi'|^2 + |\chi'|^2 \simeq |\phi'|^2 + \mathcal{O}\left(\frac{1}{c^2}\right)$$

$\Downarrow$

exactly what expected from Schrödinger eq.

2. Move on to $\mathcal{O}\left(\frac{1}{c^2}\right)$

Start with prob. density: we keep $|\chi'|^2$

$$\rightarrow \rho = |\phi'|^2 + \frac{\hbar^2}{4m^2c^2} |\vec{\sigma} \cdot \vec{\nabla} \phi'|^2$$

$\Downarrow$

not in Schrödinger form

But to write the Schrödinger eq. corresponding to the NR limit of Dirac eq., we need a spinor ϕ_{Sch} such that

$$\int d^3x \, |\phi_{Sch}|^2 \equiv \int d^3x \left[|\phi'|^2 + \frac{\hbar^2}{4m^2c^2} |\vec{\sigma}\cdot\vec{\nabla}\phi'|^2 \right]$$

Integrate by parts (symm. way)

$$\int d^3x \left(\partial_i \phi'^{\dagger} \sigma_i\right)\left(\sigma_k \partial_k \phi'\right) = -\frac{1}{2}\left[\int d^3x \, \phi'^{\dagger} \sigma_i \sigma_k \partial_i \partial_k \phi' + \int d^3x \, \partial_i \partial_k \phi'^{\dagger} \sigma_i \sigma_k \phi'\right]$$

$$\sigma_i \sigma_k = \delta_{ik} + i\epsilon_{ikm}\sigma_m$$

$$= -\frac{1}{2}\int d^3x \left[\phi'^{\dagger} \vec{\nabla}^2 \phi + \vec{\nabla}^2 \phi'^{\dagger} \phi'\right]$$

Thus

$$\int d^3x\, |\phi_{sch}|^2 = \int d^3x \left[|\phi'|^2 - \frac{\hbar^2}{8m^2c^2}\, \phi'^{\dagger}\vec{\nabla}^2\phi' - \frac{\hbar^2}{8m^2c^2}\, \vec{\nabla}^2\phi'^{\dagger}\phi' \right]$$

$$= \int d^3x \left| \phi' - \frac{\hbar^2}{8m^2c^2}\vec{\nabla}^2\phi' \right|^2 + \mathcal{O}\left(\frac{1}{c^4}\right)$$

$\Rightarrow$ we obtain

$$\phi_{sch} \equiv \left(1 + \frac{\hat{\vec{P}}^2}{8m^2c^2}\right)\phi'$$

We also need to consider terms of $\mathcal{O}\left(\frac{1}{c^2}\right)$ in the Dirac eq.: take ϕ' & χ' = stationary with energy $\mathcal{E}$

Eq. for χ':

$$\left(1 + \frac{\mathcal{E}}{2mc^2}\right)\chi' = \frac{\vec{\sigma}\cdot\hat{\vec{p}}\,\phi'}{2mc}$$

$$\Downarrow$$

$$\chi' \simeq \left(1 - \frac{\mathcal{E}}{2mc^2}\right)\frac{\vec{\sigma}\cdot\hat{\vec{p}}\,\phi'}{2mc}$$

Eq. for ϕ': $\quad \varepsilon\phi' = c\,\vec{\sigma}\cdot\hat{\vec{p}}\,\chi'$

$$\varepsilon\left(1-\frac{\hat{\vec{p}}^{\,2}}{8m^2c^2}\right)\phi_{sch} = \cancel{c}\,\vec{\sigma}\cdot\hat{\vec{p}}\left(1-\frac{\varepsilon}{2mc^2}\right)\frac{\vec{\sigma}\cdot\hat{\vec{p}}}{2m\cancel{c}}\left(1-\frac{\hat{\vec{p}}^{\,2}}{8m^2c^2}\right)\phi_{sch}$$

$$\varepsilon\left(1-\frac{\hat{\vec{p}}^{\,2}}{8m^2c^2}\right)\phi_{sch} = \frac{\hat{\vec{p}}^{\,2}}{2m}\left(1-\frac{\varepsilon}{2mc^2}\right)\left(1-\frac{\hat{\vec{p}}^{\,2}}{8m^2c^2}\right)\phi_{sch}$$

$$= \frac{\hat{\vec{p}}^{\,2}}{2m}\left(1-\frac{\hat{\vec{p}}^{\,2}}{8m^2c^2}\right)\phi_{sch}$$

$$-\frac{\varepsilon}{2mc^2}\frac{\hat{\vec{p}}^{\,2}}{2m}\phi_{sch} + \mathcal{O}\left(\frac{1}{c^4}\right)$$

$$\varepsilon\left(1+\frac{\hat{\vec{p}}^{\,2}}{8m^2c^2}\right)\phi_{sch} = \frac{\hat{\vec{p}}^{\,2}}{2m}\left(1-\frac{\hat{\vec{p}}^{\,2}}{8m^2c^2}\right)\phi_{sch}$$

$$\varepsilon\phi_{sch} = \frac{\hat{\vec{p}}^{\,2}}{2m}\left(1-\frac{\hat{\vec{p}}^{\,2}}{8m^2c^2}\right)^2\phi_{sch}$$

$$\Rightarrow \mathcal{E}\,\phi_{sch} = \left[\frac{\hat{\vec{P}}^2}{2m} - \frac{\hat{\vec{P}}^4}{8m^2c^2}\right]\phi_{sch} + \mathcal{O}\left(\frac{1}{c^4}\right)$$

$\Uparrow$

EXACTLY THE NR EXPANSION OF THE KINETIC ENERGY!

NR Dirac equation in EM field

Taking the NR limit of the Dirac eq. in an EM background we'll obtain the relativistic corrections that must be applied e.g. to atoms $\rightarrow$ <u>FINE STRUCTURE</u>

EM background $\rightarrow$ $\begin{cases} \mathcal{E} \rightarrow E \equiv \mathcal{E} - e\Phi \\ \hat{\vec{p}} \rightarrow \hat{\vec{Q}} \equiv \hat{\vec{p}} - \frac{e}{c}\vec{A} \end{cases}$

Take again ϕ' & χ' = stationary states

$\Rightarrow$ Dirac eq. for χ' becomes

$$\boxed{\chi' \simeq \frac{1}{2mc}\left(1 - \frac{E}{2mc^2}\right)\vec{\sigma}\cdot\hat{\vec{Q}}\,\phi'}$$

Keeping terms up to $\mathcal{O}\left(\frac{1}{c^2}\right)$:

$$\chi' \simeq \frac{1}{2mc}\left(1 - \frac{\mathcal{E} - e\Phi}{2mc^2}\right)\vec{\sigma}\cdot\left(\hat{\vec{p}} - \frac{e}{c}\vec{A}\right)\phi'$$

$$\simeq \frac{1}{2mc}\vec{\sigma}\cdot\hat{\vec{p}}\,\phi' - \frac{e}{2mc^2}\vec{\sigma}\cdot\vec{A}\,\phi'$$

$$\simeq \frac{1}{2mc}\vec{\sigma}\cdot\hat{\vec{Q}}\,\phi' + \mathcal{O}\left(\frac{1}{c^3}\right)$$

Inserting in the probability density we get

$$\rho \simeq |\phi'|^2 + \frac{\hbar^2}{4m^2c^2}\left|\vec{\nabla}\phi'\right|^2 \Rightarrow \text{as before!}$$

(the term with $\vec{A}$ contributes $\mathcal{O}\left(\frac{1}{c^3}\right)$)

Still true that

$$\phi_{sch} \simeq \left(1 + \frac{\hat{\vec{p}}^2}{8m^2c^2}\right)\phi'$$

Eq. for ϕ' becomes

$$E\phi' = c\,\vec{\sigma}\cdot\hat{\vec{Q}}\,\chi'$$

$$\simeq \frac{1}{2m}\vec{\sigma}\cdot\hat{\vec{Q}}\left(1-\frac{E}{2mc^2}\right)\vec{\sigma}\cdot\hat{\vec{Q}}\,\phi'$$

$$\simeq \frac{1}{2m}\left(\vec{\sigma}\cdot\hat{\vec{Q}}\right)^2\phi' - \frac{1}{4m^2c^2}\vec{\sigma}\cdot\hat{\vec{Q}}\,E\,\vec{\sigma}\cdot\hat{\vec{Q}}\,\phi'$$

cannot pass through because now it is a function, and $\hat{\vec{p}}$ acts as a derivative on functions

Computation various terms:

$$(\vec{\sigma}\cdot\hat{\vec{Q}})^2\phi' = \left[\vec{\sigma}\cdot\left(\hat{\vec{p}}-\frac{e}{c}\vec{A}\right)\right]^2\phi'$$

$$= \sigma_i\sigma_j\left(\hat{p}_i-\frac{e}{c}A_i\right)\left(\hat{p}_j-\frac{e}{c}A_j\right)\phi'$$

$$= \left(\delta_{ij}+i\epsilon_{ijk}\sigma_k\right)\left(\hat{p}_i-\frac{e}{c}A_i\right)\left(\hat{p}_j-\frac{e}{c}A_j\right)\phi'$$

$$= \left(\hat{\vec{p}}-\frac{e}{c}\vec{A}\right)^2\phi' + i\epsilon_{ijk}\sigma_k\Big(\overbrace{\hat{p}_i\hat{p}_j}^{\text{sym}} - \frac{e}{c}A_i\hat{p}_j - \frac{e}{c}\hat{p}_iA_j + \frac{e^2}{c^2}\underbrace{A_iA_j}_{\text{sym}}\Big)\phi'$$

$$= \left(\hat{\vec{p}}-\frac{e}{c}\vec{A}\right)^2\phi' + i\epsilon_{ijk}\sigma_k\left(-\frac{e}{c}A_i\hat{p}_j - \frac{e}{c}(\hat{p}_iA_j) - \frac{e}{c}A_j\hat{p}_i\right)\phi'$$

symm

$$= \left(\hat{\vec{p}}-\frac{e}{c}\vec{A}\right)^2\phi' - \frac{e\hbar}{c}(\vec{\nabla}\wedge\vec{A})\cdot\vec{\sigma}\,\phi'$$

$$= \left(\hat{\vec{p}} - \frac{e}{c}\vec{A}\right)^2 \phi' - \frac{e\hbar}{c}\vec{B}\cdot\vec{\sigma}\,\phi'$$

$\vec{\sigma}\cdot\hat{\vec{Q}}\; E\; \vec{\sigma}\cdot\hat{\vec{Q}}\,\phi' =$ already multiplied by $\frac{1}{c^2}$

(see pg. 90) $\Rightarrow$ keep $\hat{\vec{Q}} \to \hat{\vec{p}}$

$$= \vec{\sigma}\cdot\hat{\vec{p}}\,(\mathcal{E} - e\Phi)\,\vec{\sigma}\cdot\hat{\vec{p}}\,\phi'$$

$$= \sigma_i\sigma_j\;\hat{p}_i\,(\mathcal{E} - e\Phi)\,\hat{p}_j\,\phi'$$

$$= (\delta_{ij} + i\epsilon_{ijk}\sigma_k)(-i\hbar)^2$$

$$\partial_i\,(\mathcal{E} - e\Phi)\,\partial_j\,\phi'$$

acts on everything on its right

$$= -\hbar^2(\delta_{ij} + i\epsilon_{ijk}\sigma_k)$$

$$\left[-e(\partial_i\Phi)\partial_j\phi' + (\mathcal{E} - e\Phi)\partial_i\partial_j\phi'\right]$$

$$= -\hbar^2\left[-e(\vec{\nabla}\Phi)\cdot\vec{\nabla}\phi' + ie\left((\vec{\nabla}\Phi)\wedge\vec{\nabla}\phi'\right)\cdot\vec{\sigma} + (\mathcal{E}-e\Phi)\vec{\nabla}^2\phi'\right]$$

use $\vec{\nabla}\Phi \equiv -\vec{E}$ ⊕ $\vec{\nabla} = \frac{i}{\hbar}\hat{\vec{p}}$

$$= -ie\hbar\,\vec{E}\cdot\hat{\vec{p}}\,\phi' + \hbar e(\vec{E}\wedge\hat{\vec{p}})\cdot\vec{\sigma}\,\phi' + (\mathcal{E}-e\Phi)\hat{\vec{p}}^2\phi'$$

Back to pg. 90 :

$$E\phi' = \frac{1}{2m}(\vec{\sigma}\cdot\hat{\vec{Q}})^2\phi' - \frac{1}{4m^2c^2}\vec{\sigma}\cdot\hat{\vec{Q}}\,E\,\vec{\sigma}\cdot\hat{\vec{Q}}\,\phi'$$

$$= \frac{1}{2m}\left(\hat{\vec{p}} - \frac{e}{c}\vec{A}\right)^2\phi' - \frac{e\hbar}{2mc}\vec{B}\cdot\vec{\sigma}\,\phi'$$

$$- \frac{1}{4m^2c^2}\left[-ie\hbar\,\vec{E}\cdot\hat{\vec{p}}\,\phi' + \hbar e(\vec{E}\wedge\hat{\vec{p}})\cdot\vec{\sigma}\,\phi' + \underbrace{(\mathcal{E} - e\Phi)\hat{\vec{p}}^2\phi'}_{} \right]$$

LHS

$$\mathcal{E}\left(1 + \frac{\hat{\vec{p}}^2}{4m^2c^2}\right)\phi' = \left[\frac{\left(\hat{\vec{p}} - \frac{e}{c}\vec{A}\right)^2}{2m} - \frac{e\hbar}{2mc}\vec{B}\cdot\vec{\sigma} + \frac{ie\hbar}{4m^2c^2}\vec{E}\cdot\hat{\vec{p}} - \frac{e\hbar}{4m^2c^2}(\vec{E}\wedge\hat{\vec{p}})\cdot\vec{\sigma} + e\Phi - \frac{e}{4m^2c^2}\Phi\hat{\vec{p}}^2\right]\phi'$$

call $\hat{H}'$

Hamiltonian applied on ϕ_{sch}

$$\varepsilon \left(1+\frac{\hat{\vec{p}}^2}{4m^2c^2}\right)\left(1-\frac{\hat{\vec{p}}^2}{8m^2c^2}\right)\phi_{sch} = \hat{H}'\left(1-\frac{\hat{\vec{p}}^2}{8m^2c^2}\right)\phi_{sch}$$

$$\Downarrow$$

$$\varepsilon\,\phi_{sch} = \hat{H}\,\phi_{sch}$$ with

$$\begin{aligned}\hat{H} &= \left(1+\frac{\hat{\vec{p}}^2}{8m^2c^2}\right)\left(1-\frac{\hat{\vec{p}}^2}{4m^2c^2}\right)\hat{H}'\left(1-\frac{\hat{\vec{p}}^2}{8m^2c^2}\right)\\ &= \left(1-\frac{\hat{\vec{p}}^2}{8m^2c^2}\right)\hat{H}'\left(1-\frac{\hat{\vec{p}}^2}{8m^2c^2}\right)\\ &= \hat{H}' - \frac{1}{8m^2c^2}\left(\hat{\vec{p}}^2\hat{H}' + \hat{H}'\hat{\vec{p}}^2\right) + \mathcal{O}\left(\frac{1}{c^4}\right)\end{aligned}$$

only terms of $\mathcal{O}\left(\frac{1}{c^0}\right)$ are

$$\hat{H}' \rightarrow \frac{\hat{\vec{p}}^2}{2m} + e\Phi$$

$$= \hat{H}' - \frac{1}{8m^2c^2}\frac{\hat{\vec{p}}^4}{m} - \frac{e}{8m^2c^2}\underbrace{\left(\hat{\vec{p}}^2\Phi + \Phi\hat{\vec{p}}^2\right)}$$

need to properly open the derivatives:

$$\hat{\vec{p}}^2\Phi = \hat{\vec{p}}\cdot\hat{\vec{p}}\Phi = \hat{\vec{p}}\cdot\left[(\hat{\vec{p}}\Phi) + \Phi\hat{\vec{p}}\right]$$

$$= (\hat{\vec{p}}^2\Phi) + 2(\hat{\vec{p}}\cdot\Phi)\cdot\hat{\vec{p}} + \Phi\hat{\vec{p}}^2$$

where
$$\begin{cases} \hat{\vec{p}}^2\Phi = -\hbar^2\vec{\nabla}^2\Phi = \hbar^2\vec{\nabla}\cdot\vec{E} \\ \hat{\vec{p}}\Phi = -i\hbar\vec{\nabla}\Phi = +i\hbar\vec{E} \end{cases}$$

Putting all together:

$$\boxed{\begin{aligned}\hat{H} = \frac{1}{2m}\left(\hat{\vec{p}} - e\vec{A}\right)^2 - \frac{\hat{\vec{p}}^4}{8m^3c^2} - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} \\ + \frac{ie\hbar}{4m^2c^2}\vec{E}\cdot\hat{\vec{p}} - \frac{e\hbar}{4m^2c^2}\left(\vec{E}\wedge\hat{\vec{p}}\right)\cdot\vec{\sigma} \\ + e\Phi - \frac{e\hbar^2}{8m^2c^2}\left(\vec{\nabla}\cdot\vec{E}\right)\end{aligned}}$$

CLASSIFICATION OF TERMS

- $-\dfrac{\hat{\vec{p}}^4}{8m^3c^2} \rightarrow$ Correction kinetic energy

- $-\dfrac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} \rightarrow$ magnetic dipole term

$$\hat{\vec{\mu}} = \frac{e\hbar}{2mc}\vec{\sigma} = \frac{e\hbar}{mc}\hat{\vec{S}}$$

- $-\frac{e\hbar}{4m^2c^2}(\vec{E}\wedge\hat{\vec{p}})\cdot\vec{\sigma} \longrightarrow$ spin-orbit interaction

$$\|$$

$$-\frac{e\hbar}{2m^2c^2}(\vec{E}\wedge\hat{\vec{p}})\cdot\hat{\vec{S}}$$

$\downarrow$

Why? Take $\vec{E} = -\frac{dV}{dr}\hat{e}_r$

(central field in static case)

$\Downarrow$

$$\vec{E}\wedge\hat{\vec{p}} = -\frac{dV}{dr}\frac{1}{r}\hat{\vec{r}}\wedge\hat{\vec{p}}$$

$$= -\frac{dV}{dr}\frac{1}{r}\hat{\vec{L}}$$

$\swarrow$

$$+\frac{e\hbar}{2m^2c^2}\frac{dV}{dr}\frac{1}{r}\hat{\vec{L}}\cdot\hat{\vec{S}}$$

- $-\frac{e\hbar^2}{8m^2c^2}\underbrace{(\vec{\nabla}\cdot\vec{E})}_{\rho_{charge}} \rightarrow$ Darwin term

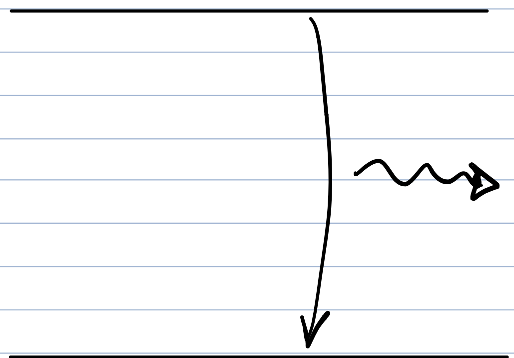
DRAWBACKS OF THE DIRAC EQUATION

Summary of wave eqs.

	Relativistic	$\rho = \|\psi\|^2$ conserved	Negative energies
Schrödinger	X	V	X
K-G	V	X	V
Dirac	V	V	V

Dirac eq: we have a relativistic wave equation which has a conserved probability density (consequence of being a diff. eq. 1^{st} order in time), but the price to pay are states with negative energies

physically a disaster



$E>0$

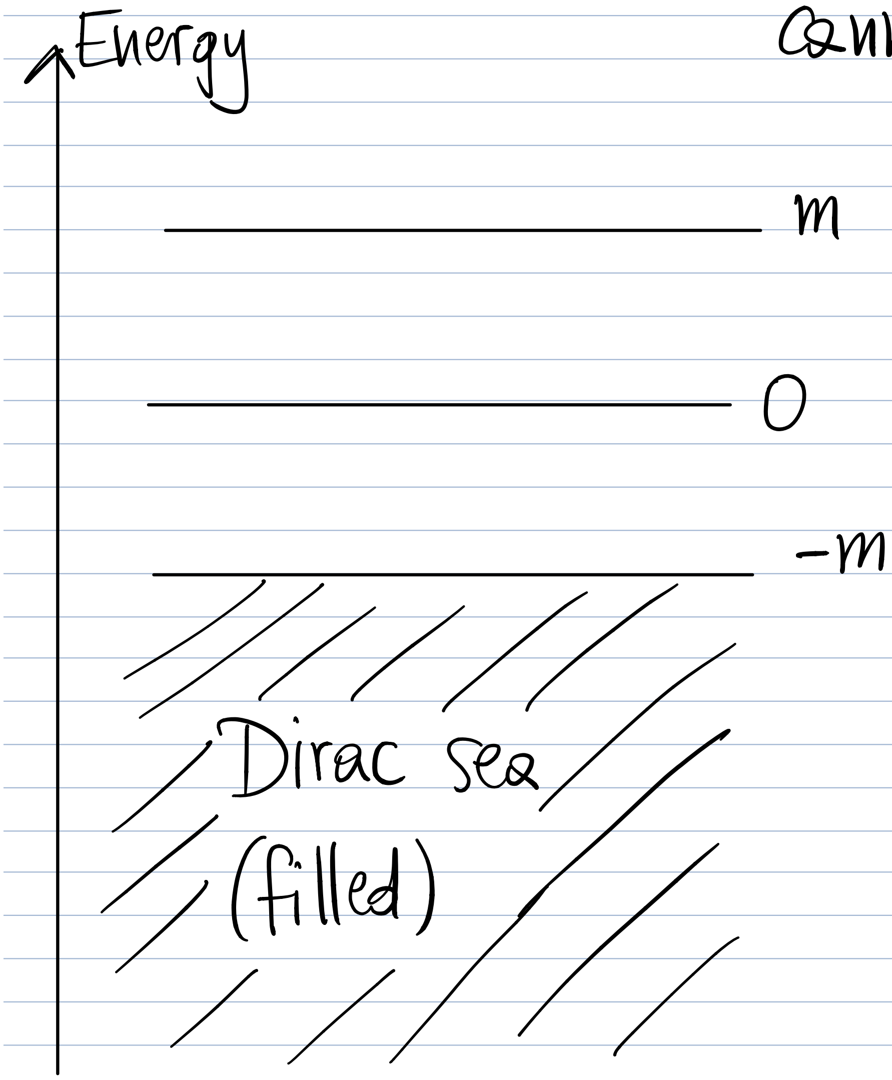
photon emission $\Rightarrow$ states with $E>0$ can decay in states with $E<0$

$E<0$

Dirac's proposal

Dirac sea : ALL STATES WITH $E<0$ ARE FILLED UP AND OBEY PAULI PRINCIPLE

$\Rightarrow$ states with $E>0$ cannot decay



Take aways :

1. there should be a spin-statistic connection (fermions = obey Pauli principle & satisfy Dirac eq $\Rightarrow$ $S = 1/2$)

2. Dirac eq. does not make sense as a theory of a single particle (physics is forcing us to introduce the Dirac sea to make the eq. feasible)

$$\Downarrow$$

WE NEED A RELATIVISTIC MANY PARTICLE THEORY

$$\Downarrow$$

WILL BE QUANTUM FIELD THEORY