Advanced Quantum Mechanics
– Exercises –
(2019)

Contents

1 Preliminaries

We list here some exercises that the student should be able to solve after any basic course on Quantum mechanics.

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1.1. Consider the time evolution operator

$$|\alpha(t)\rangle = \hat{U}(t,t')\,|\alpha(t')\rangle \;.$$

   (a) Derive the differential equation obeyed by $\hat{U}(t,t')$;

   (b) Derive an integral equation for $\hat{U}(t,t')$ equivalent to the previous differential equation.

1.2. Show that the Schrödinger and Heisenberg pictures give the same mean value for *any* observable $\hat{O}$.

1.3. Show that the Equations of Motion for and operator $\hat{A}_H(t)$ in the Heisenberg picture is

$$i\hbar\frac{d\hat{A}_H(t)}{dt} = \left[\hat{A}_H(t),\hat{H}\right] \;.$$

1.4. Verify explicitly that the differential equation

$$P(\partial_x)\phi(\boldsymbol{x}) = j(\boldsymbol{x})\,,$$

where $P(\partial_x)$ is some polynomial of partial derivatives, admits the solution

$$\phi(\boldsymbol{x}) = \phi_0(\boldsymbol{x}) + \int d^3y\; G(\boldsymbol{x}-\boldsymbol{y})j(\boldsymbol{y})\,,$$

where $G(\boldsymbol{x}-\boldsymbol{y})$ is the Green's function, defined by

$$P(\partial_x)G(\boldsymbol{x}-\boldsymbol{y}) = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\,.$$

1.5. Show that the propagator $K(\boldsymbol{x_a},t_a|\boldsymbol{x_b},t_b) \equiv K(a|b)$ is the Green's function of the Schrödinger equation. It can be useful to first show that

$$\langle \boldsymbol{x'}|\,\hat{\boldsymbol{P}}\,|\boldsymbol{x}\rangle = -i\hbar\boldsymbol{\nabla}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x'})\,,$$

and that

$$\langle \boldsymbol{x'}|\,\hat{H}\,|\boldsymbol{x}\rangle = \left[\frac{(-i\hbar)^2}{2m}\boldsymbol{\nabla}^2 + V(\boldsymbol{x})\right]\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x'})\,.$$

2 Symmetries

2.1. (a) Show that for antiunitary operators we must have ($\alpha \in \hat{C}$)

$$|\hat{U}(\alpha\psi)\rangle = \alpha^* \, |\hat{U}\psi\rangle \; .$$

(b) Show that if $\hat{U}_1$ and $\hat{U}_2$ are symmetries then so is $\hat{U} = \hat{U}_1\hat{U}_2$.

(c) Show that if $\hat{U}$ is a symmetry then so is $\hat{U}^{-1}$.

2.2. Show that the operator identity

$$\lim_{N\to\infty} \left(\mathbb{1} + \frac{i\theta}{N}\hat{T} \right)^N = e^{i\theta\hat{T}}$$

is true for a selfadjoint operator (hint: since $\hat{T}$ admits a complete set of eigenvectors, one can write...).

2.3. Show that a similarity transformation $\hat{A} \to \hat{U}^\dagger \hat{A} \hat{U}$ does not change the eigenvalues of $\hat{A}$.

2.4. Consider the translation operator

$$\hat{U}(\boldsymbol{a}) = e^{-i\boldsymbol{a}\cdot\vec{\hat{P}}/\hbar} \, .$$

Show that $[\hat{P}_i, \hat{P}_j] = 0$ is equivalent to

$$\hat{U}(\boldsymbol{b})\hat{U}(\boldsymbol{a}) = \hat{U}(\boldsymbol{a})\hat{U}(\boldsymbol{b}) \, .$$

2.5. (a) Write a rotation as $R \simeq \mathbb{1} + \omega$. What is the condition that ω must satisfy for R to be an orthogonal matrix?

(b) Write explicitly the ω matrix for the $SO(3)$ group and, by generalization, write the ω matrix for the $SO(4)$ group;

(c) How many parameters does ω contain?

(d) Writing now

$$\hat{U}(\mathbb{1} + \omega) = \mathbb{1} - \frac{i}{2}\omega_{km}\hat{J}_{km}$$

(why the factor 1/2?), and using the property of vector observables

$$\hat{U}^\dagger(\mathbb{1}+\omega)\,\hat{V}_i\,\hat{U}(\mathbb{1}+\omega) = R_{ij}\hat{V}_j\,,$$

show that

$$i\left[\hat{V}_k, \hat{J}_{ij}\right] = \delta_{ik}\hat{V}_j - \delta_{kj}\hat{V}_i\,.$$

(e) Consider now two different rotations R and R'. Using

$$\hat{U}^\dagger(R')\hat{U}(\mathbb{1}+\omega)\hat{U}(R') = U[R'^\dagger(\mathbb{1}+\omega)R']\,,$$

and expanding the expression up to first order in ω, show that $\hat{J}_{ij}$ is a tensor operator;

(f) Writing now $R' = \mathbb{1}+\omega'$, compute

$$\frac{\left[\hat{J}_{ij}, \hat{J}_{km}\right]}{i} =?$$

(g) Show that in the $\mathbb{R}^3$ case we obtain the correct $SO(3)$ algebra with the identification

$$\hat{J}_K \equiv \frac{1}{2}\varepsilon_{ijk}\hat{J}_{ij}\,.$$

2.6. (a) Given a vector V_i, $i = 1, \ldots, n$, we define n gamma matrices such that

$$\hat{V} = V_i\gamma_i\,.$$

Requiring now $\hat{V}^2 = (V_iV_i)\mathbb{1}$, what is the condition that the γ matrices must satisfy? This condition defines a *Clifford algebra*;

(b) Assuming that a rotation can be described by

$$\hat{V} \to \Omega^\dagger\hat{V}\Omega\,,$$

is the condition $\hat{V}^2 = (V_iV_i)\mathbb{1}$ still true? This transformation defines the *spinorial* representation of $SO(N)$;

(c) Writing now

$$\Omega = \mathbb{1} + \frac{i}{2}\omega_{ik}\,\hat{J}^S_{ik}\,,$$

(where $\hat{J}^S_{ik}$ is the generator of the spinorial representation), show that if we identify

$$\hat{J}^S_{ik} = \frac{i}{4}\left[\gamma_i, \gamma_k\right]\,,$$

we obtain the correct $SO(N)$ algebra;

(d) Show that in the $SO(3)$ case we have

$$\hat{J}^S_i = \frac{\sigma_i}{2}\,;$$

(e) Using tensor products between $\boldsymbol{\sigma}$ and $\mathbb{1}$, construct the γ matrices in the $SO(4)$ case.

2.7. Show that even (odd) wave functions are even (odd) parity eigenstates;

2.8. Consider a particle subject to the potential $V = V_0 \sin(2\pi x/a)$:

(a) What are the symmetries of the Hamiltonian $\hat{H}$?

(b) Is momentum conserved?

3 Identical particles and second quantization

3.1. Verify explicitly that the operators

$$\hat{S} \equiv \frac{\mathbb{1} + \hat{P}_{12}}{2}, \quad \hat{A} \equiv \frac{\mathbb{1} - \hat{P}_{12}}{2},$$

(where $\hat{P}_{12}$ is the permutation operator) are projectors.

3.2. Prove that, given any operator $\hat{O}$ acting on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$,

$$\hat{S}\hat{O}\hat{S} = \frac{\hat{S}\hat{O} + \hat{O}\hat{S}}{2}.$$

3.3. Consider a 3 particles state $|\lambda_1\lambda_2\lambda_3\rangle$, where λ_i denote a generic set of quantum numbers:

(a) what are the symmetrizer/antisymmetrizer in this case?

(b) write down the completely symmetric state.

3.4. Consider a system of two spin 1/2 particles. In each of the single particle Hilbert space we can define a basis according to

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) What is the dimension of the tensor space $\mathcal{H}_1 \otimes \mathcal{H}_2$?

(b) Construct explicitly in this space the basis $|{+}{+}\rangle \equiv |+\rangle_1 \otimes |+\rangle_2$, $|{-}{-}\rangle$, $|{+}{-}\rangle$ and $|{-}{+}\rangle$;

(c) Construct the permutation operator $\hat{P}_{12}$;

(d) Construct explicitly the symmetrizer $\hat{S}$ and the antisymmetrizer $\hat{A}$ in this space;

(e) Verify that the operators $\hat{S}$ and $\hat{A}$ computed above are projectors.

3.5. Repeat Exercise 4 above for the case of a system of three spin 1/2 particles (suggestion: use Wolfram Mathematica to speed up the computations). Show in particular that, unlike what happens with two particles, we now have

$$\mathbb{1} \neq \hat{S} + \hat{A}.$$

3.6. Two identical electrons have momenta $\boldsymbol{p_1}$ and $\boldsymbol{p_2}$, respectively. Write down the state of the system if the total spin is 0.

3.7. Consider a pair of electrons constrained to move in one dimension, in a total spin $S = 1$ state. The electrons interact through the potential

$$V(x_1, x_2) = \begin{cases} 0 & |x_1 - x_2| > a, \\ -V_0 & |x_1 - x_2| \le a. \end{cases}$$

Find the lowest energy solution with vanishing total momentum. (Hint: it is useful to go to the center of mass frame.)

3.8. The path integral treatment of identical particles show that the relevant basis in position space is

$$|\boldsymbol{x_1 x_2}\rangle_{S,A} \equiv \frac{|\boldsymbol{x_1 x_2}\rangle \pm |\boldsymbol{x_2 x_1}\rangle}{\sqrt{2}} \,.$$

Show explicitly that for $|\psi\rangle = |\psi_1 \psi_2\rangle$ one obtains automatically symmetric/antisymmetric wave functions for the system.

3.9. Suppose we consider the diatomic molecule composed by two ^{56}Fe atoms (which has $s = 0$). What is the state of the system? Repeat the same analysis in the case in which the molecule is composed by two ^{55}Fe atoms (with $s = 3/2$).

3.10. Solve the eigenvalue problem for the Helium atom using perturbation theory (with perturbation $\hat{H}_1 = 1/(4\pi\varepsilon_0 r_{12})$) and compare the result with what is obtained using the variational method.

3.11. Show explicitly that using second quantization techniques, we get

$$\langle \boldsymbol{p_1 p_2} | \boldsymbol{p_3 p_4} \rangle = \delta^{(3)}(\boldsymbol{p_1} - \boldsymbol{p_3})\delta^{(3)}(\boldsymbol{p_2} - \boldsymbol{p_4}) \pm \delta^{(3)}(\boldsymbol{p_1} - \boldsymbol{p_4})\delta^{(3)}(\boldsymbol{p_2} - \boldsymbol{p_3}) \,.$$

3.12. Consider a complete set of eigenvectors $\{|\lambda_i\rangle\}$. What is the second quantization representation in this basis of a single-particle operator $\hat{O}_1$ and of a two-particle operator $\hat{O}_2$?

3.13. Using the results of the previous exercise, write down the second quantization representation of the spin operator $\hat{\boldsymbol{S}} = \boldsymbol{\sigma}/2$.

3.14. Consider a generic Hamiltonian

$$\hat{H} = \frac{\hat{\boldsymbol{p}}}{2m} + V(\hat{\boldsymbol{r}})\,.$$

(a) write down the momentum representation (in terms of the operators $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$);

(b) write down the position representation (in terms of the field operators $\hat{\psi}(\boldsymbol{x})$ and $\hat{\psi}^\dagger(\boldsymbol{x})$).

3.15. How can the Coulomb potential be written in terms of the field operators $\hat{\psi}(\boldsymbol{x})$ and $\hat{\psi}^\dagger(\boldsymbol{x})$?

4 State operator and quantum statistical mechanics

4.1. Prove the following properties of the state operator:

(a) $\hat{\rho}$ is hermitian;

(b) $\mathrm{tr}(\hat{\rho}) = 1$;

(c) if $\hat{\rho}^2 = \hat{\rho}$ then the state is pure (*i.e.* there is only one state, appearing with probability 1);

(d) $\hat{\rho}$ is positive: $\langle\phi|\,\hat{\rho}\,|\phi\rangle \geq 0\ \forall\phi$;

4.2. Consider light with circular polarization (it can be either left or right polarization):

(a) Write the expression of the classical electric field;

(b) Translate the previous result in terms of a density state for circular polarization;

(c) Consider now a polarimeter with axis along the $(\cos\xi, \sin\xi)$ direction. What is the probability of measuring light with polarization along this direction after the light passes through the polarimeter?

(d) What changes in the previous point if the initial beam is composed by 70% left polarized light (and 30% right polarized light)?

4.3. The ammonia molecule $\mathrm{NH_3}$ is a typical quantum two level system, in which the N atom can be above the plane identified by the three H atoms (state $|\uparrow\rangle$) or below the plane (state $|\downarrow\rangle$). The Hamiltonian describing the NH_3 molecule is

$$\hat{H} = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix} .$$

(a) Suppose we prepare an initial ammonia state $|\uparrow\rangle$. What is the density matrix associated with this state?

(b) Compute the time evolution of the density matrix of the previous item.

4.4. Consider the number density of particles $\rho(\boldsymbol{x}) = \sum_{i=1}^{N} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x_i})$, describing N particles at positions $\{\boldsymbol{x_i}\}$. Express the quantum operator corresponding to the number density in terms of field operators in second quantization.

4.5. How to count states of a free particle? In principle the momentum of plane waves is a continuous variable, *i.e.* there is no notion of counting. It will however prove useful in various situations to define a density of states in order to count how many states have momentum around a certain fixed value. In order to count these states, we put our system in a large cubic box of size L imposing periodic boundary conditions.

(a) What is the condition that the momentum must satisfy once periodic boundary conditions are imposed?

(b) How many states to we have in a volume $\Delta p_x \Delta p_y \Delta p_z$ in momentum space if the particle is a boson? And if it is a fermion?

(c) How many fermions to fill up completely the lowest energy cell (starting with $\boldsymbol{p} = 0$)? What is the total energy of the system?

4.6. A system of N conductivity electrons in a metal can be considered as almost free. Using the results of the previous exercise, compute:

(a) The radius of the sphere in momentum space that contains exactly N electrons (it is an excellent approximation to consider L sufficiently large to use integrals to sum over states). This momentum is called the **Fermi momentum** and is usually denoted by p_F;

(b) The energy E_F corresponding to the Fermi momentum;

(c) The energy density $g(E)$, defined as

$$N = \int_0^{E_F} dE\, g(E)\,.$$

5 Relativistic quantum mechanics

5.1. A convenient choice of units to work with in the context of relativistic quantum mechanics (and quantum field theory) is the one of *natural units*, i.e. $\hbar = 1 = c$.

(a) Show that in these units $[L] = [E]^{-1}$;

(b) What is the relation between time and energy units? Show explicitly that the result is compatible with the Schrödinger equation.

5.2. Consider a Poincaré transformation written as

$$U(\varepsilon, \mathbb{1} + \omega) \simeq \mathbb{1} + \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu} + i\varepsilon_\mu \hat{P}^\mu \,,$$

where ε and ω are infinitesimal parameters:

(a) Compute $U(a, \Lambda)U(\varepsilon, \mathbb{1} + \omega)U^{-1}(a, \Lambda)$;

(b) Using the previous result, compute

$$U(a, \Lambda)\, \hat{J}^{\mu\nu}\, U^{-1}(a, \Lambda)$$
$$U(a, \Lambda)\, \hat{P}^{\alpha}\, U^{-1}(a, \Lambda)$$

5.3. Use the previous result (specialized for infinitesimal a_μ and $\Lambda^\mu{}_\nu$) to deduce the Lie algebra of the Poincaré group.

5.4. Consider the $(1/2, 0)$ and $(0, 1/2)$ representations of the Lorentz group:

(a) show that for these representations we have

$$\left(\frac{1}{2}, 0\right) \to \boldsymbol{J} = \frac{\boldsymbol{\sigma}}{2}, \quad \boldsymbol{K} = i\frac{\boldsymbol{\sigma}}{2}\,,$$
$$\left(0, \frac{1}{2}\right) \to \boldsymbol{J} = \frac{\boldsymbol{\sigma}}{2}, \quad \boldsymbol{K} = -i\frac{\boldsymbol{\sigma}}{2}\,.$$

(b) call now ψ_L a spinor transforming in the $(1/2, 0)$ representation and ψ_R a spinor transforming in the $(0, 1/2)$ representation. Compute explicitly the form of the Lorentz transform spinors in the two cases. For simplicity, specialize to the case of infinitesimal parameters;

(c) compute the transformation properties for the hermitian conjugate spinors $\psi^\dagger_{L,R}$;

(d) is the combination $\psi^\dagger_L\psi_L$ Lorentz invariant? What about $\psi^\dagger_L\psi_R$?

(e) show that $(\psi^\dagger_R\psi_R, \psi^\dagger_R\boldsymbol{\sigma}\psi_R)$ transforms like a 4-vector;

(f) show that $(\psi^\dagger_R\psi_R, -\psi^\dagger_R\boldsymbol{\sigma}\psi_R)$ transforms like a 4-vector;

(g) is the combination $\psi^\dagger_R\partial_t\psi_R + \psi^\dagger_R\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\psi_R$ Lorentz invariant?

(h) what about the combination $\psi^\dagger_L\partial_t\psi_L + \psi^\dagger_L\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\psi_L$?

5.5. In the classical field theory of the Schrödinger equation, the Lagrangian density is

$$\mathcal{L} = \phi^\dagger\left(i\hbar\partial_t + \frac{\hbar^2\boldsymbol{\nabla}^2}{2m}\right)\phi\,,$$

where ϕ is the classical Schrödinger field.

(a) Show that the Lagrangian density can be made hermitian adding only surface terms (*i.e.* total divergencies);

(b) Compute the classical Poisson brackets between the field ϕ and its conjugate momentum;

(c) Show that the Hamiltonian computed as Laplace transform in classical Schrödinger field theory is consistent with the Hamiltonian operator obtained in second quantization (consider for simplicity only the free particle case).

5.6. The Dirac equation in natural units is $(i\gamma^\mu\partial_\mu - m)\psi = 0$. Write the Dirac equation reinstating the c and $\hbar$ factors.

5.7. Using the Lie algebra of the Poincaré group, show that once we identify the $\mu = 0$ component of the 4-momentum operator $\mathbb{P}^\mu$ with the Hamiltonian, $\mathbb{H} = \mathbb{P}^0$ we have conservation of linear and angular momentum.

5.8. Considering the classification of the Poincaré algebra in terms of $SU(2)\times SU(2)$ quantum number, what is the smallest representation that contains a spin 2 particle?

5.9. Show that $\int d^3x\psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x})$ is not a conserved quantity if a wave function satisfies the Klein-Goerdon equation.

5.10. Verify that $\int d^3x \left(i\psi^\dagger(\boldsymbol{x})\dot{\psi}(\boldsymbol{x}) - i\dot{\psi}^\dagger(\boldsymbol{x})\psi(\boldsymbol{x}) \right)$ is a conserved quantity in the Klein-Gordon theory.

6 Scattering

6.1. The flux of initial particles in a 2 particle scattering is given by

$$\Phi_{IN} = \frac{v_{IN}}{L^3},$$

where L^3 is the total volume considered and v_{IN} the relative velocity. Give an expression for v_{IN} for non relativistic and relativistic scattering.

6.2. Consider the Möller operator for a time independent Hamiltonian,

$$\hat{\Omega}(t) \equiv e^{i\hat{H}t/\hbar} e^{-i\hat{H}_0 t/\hbar}.$$

Can we write it as

$$\hat{\Omega}(t) = e^{i\hat{V}t/\hbar}?$$

6.3. Show the following results in interaction picture:

(a) differential equation for the time evolution operator: $i\hbar\partial_t \hat{U}_I(t,t') = \hat{V}_I(t)\hat{U}_I(t,t')$;

(b) Schrödinger equation: $i\hbar\partial_t |\psi_I(t)\rangle = \hat{V}_I(t)|\psi_I(t)\rangle$;

(c) evolution of the interaction (1): $\hat{V}_I(t) = \hat{U}_0(t',t)\hat{V}_I(t')\hat{U}_0(t,t')$;

(d) evolution of the interaction (2): $i\hbar\partial_t \hat{V}_I(t) = \left[\hat{V}_I(t), \hat{H}_0\right]$;

6.4. Apply the optical theorem to an initial state of two particles. What is the formula for the total cross section?

6.5. Consider the Yukawa potential

$$V(r) = \lambda \frac{e^{-mr}}{r},$$

where r is the relative distance between two particles and λ and m are constants with appropriate units.

(a) What are the units of λ and m? Is there a unit system in which they can be assigned the same units?

(b) Compute the matrix element entering the cross section in the first Born approximation;

(c) Using the previous result, compute the differential cross section of the process;

(d) Take now the $m \to 0$ limit to reduce the Yukawa potential to the Coulomb one and write the resulting differential cross section.

6.6. Consider the situation in which the interaction can be written as

$$V(\boldsymbol{x}) = \sum_{i=1}^{N} V_i(\boldsymbol{x} - \boldsymbol{x}_i)\,,$$

which represents the case of multiple scattering centers.

(a) Write the explicit expression for the scattering amplitude;

(b) Define the exchanged momentum at each scattering center as Q_i, $Q = \max(Q_i)$ and R the typical overall size of the target. What happens to the scattering amplitude when $QR \gg 1$?

(c) What happens instead in the opposite situation, $QR \ll 1$? Suppose all the individual scattering amplitudes are equal. The scattering in this situation is called "coherent scattering". In the case of neutrinos scattering off nucleus, the effect was predicted for the first time in 1974 by D. Freedman (Phys.Rev. D9 (1974) 1389-1392) and experimentally observed by the COHERENT collaboration in 2017 (Science 357 (2017) no.6356, 1123-1126).

(d) Suppose the scattering centers are nucleons inside a nucleus. Use the typical nucleus size to estimate the exchanged momentum $Q \sim 1/(4R)$ up to which we can estimate to have coherent scattering off nuclei.

6.7. As we have seen, strictly speaking the computation of the probability rate for the process $i \to f$ to happen must be computed using normalized states, *i.e.* box normalized states, if we are considering continuum eigenstates. Using the box normalization, write down:

(a) the expression for the quantized radiation field;

(b) the commutator between annihilation and creation operators;

(c) the matrix element for a dipole transition.

Check explicitly that all the dependence on the size of the cube disappears once we sum over all possible photon momenta in the final state.

6.8. Compute the matrix element for the process

$$A^* + N_{(\boldsymbol{k},\lambda)} \to A + (N+1)_{(\boldsymbol{k},\lambda)} \,,$$

where A and A^* are atomic states and $N_{(\boldsymbol{k},\lambda)}$ represents a state with N photons, all with momentum $\boldsymbol{k}$ and polarization λ. This process is called "stimulated emission".