List of exercises 9

1. Consider the hydrogen atom. We will now derive its eigenfunctions using the algebraic method. To define the notation, we will denote with $|n, \ell, m\rangle$ the eigenkets expressed in the $\left\{\hat{H}, \hat{\boldsymbol{L}}^2, \hat{L}_3\right\}$ basis, and with $|a, m_+, m_-\rangle_M$ the eigenkets in the $\left\{\hat{\boldsymbol{A}}_+^2 = \hat{\boldsymbol{A}}_-^2, \hat{A}_{+3}, \hat{A}_{-3}\right\}$ basis (with $n = 2a+1$ and $m = m_+ + m_-$).

   (a) Show that the Runge-Lenz vector operator can be written as

   $$\hat{\boldsymbol{M}} = \frac{\hat{\boldsymbol{p}} \wedge \hat{\boldsymbol{L}} - i\hbar\hat{\boldsymbol{p}}}{m} - Z^2\alpha\hbar c\frac{\hat{\boldsymbol{r}}}{\hat{r}}\,,$$

   where we have defined the fine structure constant

   $$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}\,;$$

   (b) Convince yourself that

   $$\boldsymbol{\nabla} f(r) = \boldsymbol{e}_r\frac{df(r)}{dr}\,;$$

   (c) We now consider the wave function of the ground state, $|1, 0, 0\rangle$. What is the corresponding ket in the $\left\{\hat{\boldsymbol{A}}_+^2 = \hat{\boldsymbol{A}}_-^2, \hat{A}_{+3}, \hat{A}_{-3}\right\}$ basis? Call this ket $|1\rangle$ for simplicity of notation. Convince yourself that it must be true that

   $$\hat{\boldsymbol{M}}\,|1\rangle = 0\,;$$

   (d) Using the explicit expression for the Runge-Lenz vector derived above, convert the equation $\hat{\boldsymbol{M}}\,|1\rangle = 0$ into a first order differential equation in spherical coordinate and solve it. What is the normalized eigenfunction for the ground state?

   (e) We now move to excited states, and we consider states with energy $E_2 = -13.6\,\mathrm{eV}/4$. What are the eigenkets in the $\left\{\hat{\boldsymbol{A}}_+^2 = \hat{\boldsymbol{A}}_-^2, \hat{A}_{+3}, \hat{A}_{-3}\right\}$ basis that correspond to these states?

(f) Among the states of the previous item, consider those with the highest values of ℓ and m. How many kets of the other basis correspond to these (or this) kets? Convince yourself that for these states it is true that the action of $\hat{A}^+_\alpha = (\hat{A}_{\alpha 1} + \hat{A}_{\alpha 2})/2$ (where $\alpha = \pm$) gives zero. Convert this algebraic equation to a differential equation (hint: it must be first order) and solve it. How do yo compute the wave function of the other degenerate states?

2. Consider an electron bound to a proton with wave function

$$\psi = Nre^{-r/2a_0} .$$

What are the possible outcomes of an angular momentum measurement?

3. Construct explicitly the vectors in the 6-dimensional space obtained taking the tensor product $1 \otimes 1/2$. What is the matrix form of $\hat{J}_1^2$, $\hat{J}_2^2$ and $\hat{J}_{tot}^2$?

4. Consider the effect of a uniform electric field on the excited states of a hydrogen atom (linear Stark effect):

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} - \frac{e^2}{\hat{r}} - e\hat{z}|\boldsymbol{E}|.$$

Considering the electric field as a perturbation and neglecting the fine structure effects of the hydrogen atom, determine the corrections to the eigenvectors at first order in perturbation theory for the ground state and for the first excited state. Discuss the relative degeneracy.

5. An hydrogen atom lies in a magnetic field along the z-axis. Neglecting the fine structure effects the Hamiltonian of the system is

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} - \frac{e^2}{\hat{r}} + \frac{\mu B}{\hbar}(\hat{L}_z + 2\hat{S}_z).$$

At the time $t = 0$ the electron state is given by

$$\psi(r, \theta, \phi; t = 0) = \frac{1}{4a\sqrt{4\pi a_0^3}}(4a\sqrt{2}e^{-r/a_0},\ r\,e^{-r/(2a_0)}cos\theta)^T,$$

where a_0 is the Bohr radius and we used the base spinors of the eigenstates of $\hat{S}_z$. Determine:

(a) the state at the generic time $t > 0$;

(b) the expected value, as a function of time, of $\hat{H}$, $\hat{L}^2$ and $\hat{J}^2$.

6. The state of an electron of a hydrogen atom with orbital angular momentum $\ell = 1$ and spin 1/2 is described by the normalized wave function

$$\psi(r, \phi, \theta) = \frac{R_{2,1}(r)}{\sqrt{2}} \left[Y_{1,0}(\phi, \theta)(1, 0)^T + Y_{1,1}(\phi, \theta)(0, 1)^T \right],$$

where $R_{2,1}(r)$ is the radial wavefunction for the hydrogen atom with $n = 2$ and $\ell = 1$ and $(1,\ 0)^T$ and $(0,\ 1)^T$ are the spin states $|1/2,\ \pm 1/2\rangle$.

(a) Determine the possible results of a measurement fo $\hat{J}^2$ and $\hat{J}_z$, with the respective probabilities.

(b) Compute the matrix element $\langle\psi|\hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{S}}|\psi\rangle$.

(c) Compute the possible results of a measurement of $\hat{L}_x$ and its probability. Explain why, after a measurement of $\hat{L}_x$, the state remains an egenstates of $\hat{L}^2$ and specify its eigenvalue.