# List of exercises 8

1. Consider a particle with *definite linear momentum* $\boldsymbol{p}$. What result do we obtain measuring the component of the orbital angular momentum along $\boldsymbol{p}$?

2. Consider an eigenstate of $\hat{\boldsymbol{J}}^2$ and $\hat{J}_3$. Compute the values of $\left\langle \hat{J}_1 \right\rangle$ and $\left\langle \hat{J}_2 \right\rangle$;

3. A measurement of a system in $j = 1$ along $\hat{J}_1$ gives $+\hbar$ as result. Compute the probability of getting $-\hbar$ as result of a measurement of $\hat{J}_3$;

4. Consider a classical rotator with Hamiltonian
$$H = \frac{1}{2} I \omega^2 \,,$$
where $I$ is the inertia momentum and $\omega$ the angular velocity. What are the eigenvalues and eigenvectors?

5. Consider a system with angular momentum $\ell = 1$. A measurement of the component of the angular momentum along a direction $\boldsymbol{n}$ such that $\boldsymbol{n} \cdot \boldsymbol{e}_z = \cos\theta$ gives the result $+\hbar$. What is the probability of finding the value $+\hbar$ in a measurement of $\hat{L}_3$?

6. Show that the components of the vector $\boldsymbol{x} = (x, y, z)^T$ can be written as combinations of spherical harmonics;

7. A particle is in a state with wave function
$$\psi(\boldsymbol{r}) = N\, x\, e^{-\alpha r}$$
where $N$ is a normalization factor. A measurement of $\hat{\boldsymbol{L}}^2$ is performed. What is the wave function immediately after the measurement?

8. Given the following state of a system
$$\psi(r, \theta, \phi) = A x(r)(\cos\theta + \sin\theta\cos\phi)$$
with
$$\int_0^\infty |x(r)|^2 r^2 dr = 1,$$

(a) compute $|A|$ such that $\psi$ is normalized;

(b) show that $\langle L_+\rangle = \langle L_-\rangle$;

(c) compute the time evolution and the possible results of a measurement of $L_3$ and their probability if

$$H = \frac{L^2}{2I} + \alpha L_3.$$

9. Given the following state of a system

$$\psi(r, \theta, \phi, t = 0) = R(r)(\alpha_0 Y_{1,1}(\theta, \phi) + \beta_0 Y_{1,-1}(\theta, \phi))$$

and

$$H = \frac{|\vec{L}|^2}{2I} + \alpha L_3,$$

(a) find $\psi(\vec{x}, t)$;

(b) the values of a measurement of $\vec{L}^2$, $L_3$ and $L_1$ and their probability;

(c) for which value of $t$ the probability of having $L_1 = 1$ is the largest?

(d) if we define the states

$$\psi_-(\vec{x}) = \frac{R(r)}{\sqrt{2}}(Y_{1,1} - Y_{1,-1})\psi_+(\vec{x}) = \frac{R(r)}{\sqrt{2}}(Y_{1,1} + Y_{1,-1}),$$

and the operator $F$ with the properties $F\psi_- = 0$ and $F\psi_+ = \psi_-$, find the expectation value of $F$ at a generic time $t$, $F(t) = \langle\psi|F|\psi\rangle_t$ and compute the limit

$$\lim_{T->\infty} \frac{1}{T}\int_0^T F(t)dt.$$

10. A particle, similar to a quantum spherical rotor, in a magnetic field along $y$, is described by the hamiltonian

$$H = \frac{\vec{L}^2}{2I} - gBL_2.$$

Knowing that at $t = 0$ the measurement of the orbital angular momentum gives $2\hbar^2$ and the measurement of its component along the $z-$axis gives zero, determine

(a) the state $|\psi, t\rangle$ at the generic time $t$;

(b) the expectation value on $|\psi, t\rangle$ of $H$ and $L_1$;

(c) the first time $t_0$, if it exist, such that $|\psi, t\rangle$ is an eigenstate of $L_1$.

11. Consider a spin 1/2 particle with angular momentum 1, with time evolution described by the Hamiltonian

$$H = \alpha \vec{L}^2 + \beta \vec{S}^2 + \gamma(L_3 + 2S_3).$$

(a) Determine the eigenstates relative to the total angular momentum $\vec{J}^2$ and $J_3$, where $\vec{J} = \vec{L} + \vec{S}$;

(b) Given the state of the system at $t = 0$

$$|\psi, t = 0\rangle = |3/2, 1/2, 1, 1/2\rangle_{j,j_3,l,s},$$

determine the values of measurements of $\vec{J}^2$ and $J_3$ at a generic time $t$ and their probability.

(c) If at $\bar{t}$ a measurement of $\vec{J}^2$ gives 3/4 as a result; determine what values can give a measurement of $S_3$ at a time $t > \bar{t}$ and its probability.

12. A spin 1 particle with mass m is constrained to move on the surface of a sphere of radius $R$ and lies in a magnetic field $B$ along the $z$ direction, such that its Hamiltonian is

$$H = \frac{\vec{L}^2}{2mR^2} + \frac{\mu B}{\hbar}(L_3 + 2S_3).$$

At $t = 0$ a measurement of $\vec{J}^2$ gives exactly $2\hbar^2$, a measurement of $J_3$ gives exactly $\hbar$ and one of $\vec{L}^2$ gives a result $\leq 2\hbar^2$; the probability of having $S_3 = \hbar$ is 2/3 and the expectation value of $\cos\theta$ is the largest possible. Find as a function of time

(a) the state of the system;

(b) the expectation value of $\cos\theta$ and $\vec{J}^2$.

13. Write the matrix representations for $L_3$, $L_\pm$ and $L^2$ for $\ell = 0$, 1/2 and 1.

14. A spinor of $SU(2)$ is an object that transforms in the fundamental representation, *i.e*

$$\psi_a \to U_{ab}\psi_b\,, \qquad a,b = 1,2\,,$$

where $U \in SU(2)$. In analogy to what is done for n-dimensional vectors, we can define a 2-*tensor* as an object that transforms as the product of two spinors, *i.e.*

$$T_{ab} \to U_{ac}U_{bd}T_{cd}\,.$$

(a) Show that, given two spinors $\psi$ and $\chi$, the combination

$$\chi^\dagger\psi \equiv \chi_a^*\psi_a$$

is invariant for transformations of $SU(2)$;

(b) Show that the combination

$$\varepsilon_{ab}\psi_a\chi_b$$

is also invariant under transformations of $SU(2)$.