List of exercises 7

1. Show that the Hamiltonian is the conserved Noether charge corresponding to time translations *using the Lagrangian formalism*;

2. Consider a particle subject to the potential $V = V_0 \sin(2\pi x/a)$:

   (a) What are the symmetries of the Hamiltonian $\hat{H}$?

   (b) Is momentum conserved?

3. Show that for antiunitary operators we must have ($\alpha \in \mathbb{C}$)

$$\hat{U}\left[\alpha \left|\psi\right\rangle\right] = \alpha^* \hat{U} \left|\psi\right\rangle \ .$$

4. Using the definition of symmetry in terms of probability, show that

   (a) if $\hat{U}_1$ and $\hat{U}_2$ are symmetries then so is $\hat{U} = \hat{U}_1 \hat{U}_2$.

   (b) if $\hat{U}$ is a symmetry then so is $\hat{U}^{-1}$.

5. Consider the translation operator

$$\hat{U}(\boldsymbol{a}) = e^{-i\boldsymbol{a}\cdot\hat{\vec{P}}/\hbar} \, .$$

   Show that $[\hat{P}_i, \hat{P}_j] = 0$ is equivalent to

$$\hat{U}(\boldsymbol{b})\hat{U}(\boldsymbol{a}) = \hat{U}(\boldsymbol{a})\hat{U}(\boldsymbol{b}) \, .$$

   What are the structure constants of the group generated by $\hat{\vec{P}}$?

6. Show that a simmetry transformation $\hat{A} \to \hat{U}^\dagger \hat{A} \hat{U}$ do not change the eigenvalues of $\hat{A}$;

7. Write explicitly

   (a) The generators of rotations in 4 dimensions;

   (b) The *quantum* generators of rotations in 4 dimensions (in the basis in which $\hat{J}_3$ is diagonal).

8. Compute

$$\left\langle \boldsymbol{x} \left| e^{-i\boldsymbol{a}\cdot\hat{\vec{P}}/\hbar} \right| \psi \right\rangle$$

for a generic state.

9. We have seen that the unitary operator that implements parity in Quantum Mechanics can be chosen to have a phase such that

$$\hat{\Pi}\left|\boldsymbol{x}\right\rangle = \left|-\boldsymbol{x}\right\rangle .$$

Show that, imposing well motivated conditions on the wave function $\langle \boldsymbol{x} \,|\, \boldsymbol{p} \rangle$, the same choice of phase implies that

$$\hat{\Pi}\left|\boldsymbol{p}\right\rangle = \left|-\boldsymbol{p}\right\rangle .$$

10. Consider a particle subject to the 1-dimensional potential

$$V(x) = \begin{cases} 0 & |x| < a; \\ \infty & \text{otherwise.} \end{cases}$$

(a) Show that the eigenfunctions must be either even or odd functions;

(b) Using the results of the previous item, solve the time-independent Schödinger equation and determine univocally eigenvalues and eigenfunctions.