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1 Exercises

1. Consider a particle free to move on $x = [-L/2, L/2]$ and described by the wave function:

$$\psi(x) = Ax\left(x^2 - \frac{L^2}{4}\right), \quad \text{for } |x| < \frac{L}{2}.$$

   (a) Compute the value of the coefficient A, such that the $\psi(x)$ is normalized.

   (b) Compute the time evolution of $\psi(x,t)$ and the expectation value of the energy.

   (c) Compute the probability associated to each of the possible results of a measurement of energy.

   (d) What is the value of energy with maximum probability?

2. Consider the path integral of the harmonic oscillator:

   (a) Compute the classical path between the points (x_a, t_a) and (x_b, t_b);

   (b) Using the result of the previous item, compute the classical action between times t_a and t_b;

3. A particle with mass m lies on $x = [0, L]$ and it is described at time $t = 0$ by $\psi(x, 0) = ax^2 + bx + c$. Determine

   (a) the coefficients a, b and c;

   (b) the expectation value value of energy $\langle E \rangle$;

   (c) the probability that a measurement of energy gives $E_n = \hbar^2\pi^2 n^2/(2mL^2)$;

   (d) $\psi(x, t)$.

4. Show that a coherent state

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle$$

can be written as

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger}\,|0\rangle\ .$$

5. A particle with mass m is constrained to lie on the x axis and its hamiltonian is

$$\begin{aligned} H &= \frac{p^2}{2m} + V(x), \\ V(x < -L/2) &= +\infty, \\ V(|x| < L/2) &= 0, \\ V(x > L/2) &= +\infty. \end{aligned}$$

   (a) Determine eigenvalues and eigenvectors of H.
   (b) Add the perturbation

$$\lambda \frac{\hbar^2 \pi^2}{2mL^2} \sin(\frac{\pi x}{L}).$$

   Determine the energy correction of the fundamental state at second order of perturbation theory and the second order correction to its eigenvector.

6. Consider a system with the following Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 + \alpha(x - x_0)$$

   (a) Determine exactly the eigenvalues and the eigenvectors of H.
   (b) Consider $\alpha(x - x_0)$ as a perturbation and compute at first order of perturbation theory the wave function for the ground state and at second order in perturbation theory the value of energy corresponding to the two unperturbed state with lower energy.

7. An harmonic oscillator in one dimension with mass m and frequency ω is in an initial state $|\psi, 0\rangle$ such that a measurement of the energy gives a value $E \leq 2\hbar\omega$, the expectation value of the energy is $\langle H \rangle = (3/4)\hbar\omega$ and we have that $\langle p \rangle = -m\omega\langle x \rangle < 0$. Determine the state $|\psi, t\rangle$ and the first time t_0 such that $\langle x \rangle = 0$.

8. An harmonic oscillator in one dimension with mass m and frequency ω is in an initial state $|\psi, 0\rangle$ such that a measurement of the energy gives a value E_1 with probability $3/4$ and E_2 with probability $1/4$, the expectation value of the position is zero and the expectation value of the momentum is positive. Determine the state $|\psi, t\rangle$ and the expectation values of x, p and x^2.

2 Problems

9. A particle with mass m is constrained by the potential

$$\begin{aligned} V(|x| > a) &= +\infty, \\ V(|x| < a) &= -\frac{\hbar}{2m}\gamma\delta(x). \end{aligned}$$

   (a) Determine the value of $\gamma > 0$ such that the smallest eigenvalue of H is $E = 0$ and write the normalized $\psi(x)$.

   (b) Write the equations that determine the other eigenvalues.

   (c) If $\psi(x,0) = N(q - \frac{|x|}{a} + \frac{\sin \pi x}{a})$, after what time the particle come back at the initial state? If you measure the energy, what results can you obtain and with what probability?

10. An harmonic oscillator in one dimension with mass m and frequency ω is in an initial state $|\psi, 0\rangle$ such that a measurement of the energy gives a value $E \leq 2\hbar\omega$, the expectation value of the momentum is the largest possible. Determine at $t = 0$ the state $|\psi, 0\rangle$ of the system, the expectation value of the energy and of the position. At the time t_0 when the momentum is for the first time zero, an external mechanism change the frequency to $\omega' = \omega/2$ and it adds to the Hamiltonian the constant $\hbar\omega/4$ such that the new Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{8}x^2 + \frac{\hbar\omega}{4}.$$

   Determine immediately after the change of the frequency:

   (a) the possible results of a measurement of energy;

   (b) the probability of obtaining a value of $\hbar\omega$.

11. A harmonic oscillator of mass m and frequency ω is subjected to a Delta-function wipe with velocity v,

$$H(t) = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 + \gamma\delta(x - vt)$$

   Suppose the oscillator is in the ground state in the far past, and we let the system evolve to the far future. Using first order time-dependent perturbation theory, find the probability P_n that we find the system in the $n-$th excited state after the wipe, for $n \neq 0$, and supposing that N is very large, find N such that P_N is maximum.