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1 Exercises

1. Show that

$$\langle x' \,|\, \mathcal{X} \,|\, x \rangle = x\delta(x' - x)\,.$$

2. Show explicitly that the eigenvalue problem

$$[\mathcal{P}\phi_p](x) = p\phi_p(x)$$

has as solution a plane wave. Compute it's normalization given that we want momentum eigenstates to have the same completeness structure as position eigenstates,

$$\langle p|p' \rangle = \delta(p' - p)$$

3. Compute the momentum representation of the momentum and position operators,

$$\langle p' \,|\, \mathcal{P} \,|\, p \rangle \ , \qquad \langle p' \,|\, \mathcal{X} \,|\, p \rangle \ .$$

4. Given a generic Hamiltonian

$$\mathcal{H} = \frac{\mathcal{P}^2}{2m} + V(\mathcal{X})\,,$$

compute $[\mathcal{H}\psi](p)$ for a generic state $|\psi\rangle$.

5. Show that

$$e^{a\frac{d}{dx}}\psi(x) = \psi(x + a)\,.$$

2 Problems

6. A generic Gaussian state for the free particle of mass m can be written

$$\langle x|\psi(0)\rangle = Ne^{-\frac{\alpha(x-\beta)^2}{2}},$$

where α, β are complex numbers.

(a) What is the condition on α for the state to be normalizable?

(b) Find the normalization constant N in terms of α and β.

(c) Write down the moments of the distribution, $\langle\mathcal{X}\rangle, \langle\mathcal{P}\rangle, \langle\mathcal{X}^2\rangle, \langle\mathcal{P}^2\rangle$. Verify it satisfies the uncertainty principle.

You probably already saw that going further with these calculations to solve the time-dependent version would be completely nasty and impractical. We consider instead a special case, the Gaussian wave packet starting at the origin at $t = 0$ with spread σ modulated by some momentum p_0:

$$\langle x|\psi(0)\rangle = N \exp\left[-\frac{x^2}{4\sigma^2} + i\frac{p_0 x}{\hbar}\right],$$

We'll see how this represents a particle moving with momentum p_0. You should absorb any multiplicative constants that appear into the normalization N to simplify calculations.

(d) Find the momentum representation of the state $\langle p|\psi(0)\rangle$.

(e) Write down the time evolution for the state $\langle p|\psi(t)\rangle$ in this basis. Remember that the free particle Hamiltonian

$$\mathcal{H} = \frac{\mathcal{P}^2}{2m}.$$

(f) Obtain $\langle x|\psi(t)\rangle$ by inverse transformation. You should have a Gaussian state.

(g) Find $\alpha(t)$ and $\beta(t)$ for the time evolved Gaussian wave packet. How do these, together with the moments from item (c), describe a moving particle? What observable encodes the quantum effects of the motion?