List of exercises 4

1 Exercises on the Path Integral

1. Using the propagator for the quantum free particle
$$K(b|a) = \left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{1/2} e^{\frac{im(x_b-x_a)^2}{2\hbar(t_b-t_a)}} ,$$
show that
$$\left(i\hbar\frac{\partial}{\partial t_b} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2}\right) K(b|a) = 0 ,$$
which means that the propagator obeys the same Schödinger equation as the wave function.

2. Various techniques can be used to explicitly compute the path integral, in addition to the one saw during the lectures. Here is another one:
   (a) Given an action $S[x(t), t_b, t_a] = \int_{t_a}^{t_b} dt\, L[\dot{x}(t), x(t)]$, the classical path $x_{cl}(t)$ is an extremum of the action S. Compute the differential equation that $x_{cl}(t)$ must obey (*i.e.* the equation of motion);
   (b) Each path can now be written as
   $$x(t) = x_{cl}(t) + y(t) ,$$
   i.e. we can parametrize any path measuring its "distance" from the classical path. Do you expect the path integral measure to obey $\mathcal{D}x(t) = \mathcal{D}y(t)$? What are the boundary conditions that $y(t)$ must obey at $t = t_{a,b}$?
   (c) Consider now a generic quadratic Lagrangian
   $$L = a(t)\dot{x}^2 + b(t)\dot{x}x + c(t)x^2 + d(t)\dot{x} + e(t)x + f(t) .$$
   What is the equation of motion that the classical path must obey?
   (d) Compute
   $$S[x_{cl}(t) + y(t)]$$
   and show explicitly that, using the equation of motion for $x_{cl}(t)$, the term linear in $y(t)$ vanishes;

(e) Show that the above computations lead to a kernel written in the form

$$K(b|a) = e^{iS[x_{cl}(t),t_b,t_a]/\hbar} \int_0^0 \mathcal{D}y(t) e^{\frac{i}{\hbar}\int_{t_a}^{t_b} dt\,\left(a(t)\dot{y}^2 + b(t)\dot{y}y + c(t)y^2\right)}$$
$$= e^{iS[x_{cl}(t),t_b,t_a]/\hbar} F(t_b, t_a)\,,$$

which means that for quadratic Lagrangians all the spatial dependence is given by the classical path;

(f) Confirm the previous point computing the classical path and $S[x_{cl}(t)]$ for the free particle;

(g) What about the time-dependent factor $F(t_b, t_a)$? Its computation is clearly the difficult part, since it encodes all the proper quantum effects that cannot be inferred from the classical path. It can be shown that, for the case of physical interest $a(t) = m/2$, it is given by

$$F(t_b, t_a) = \left(\frac{m}{2\pi i\hbar f(t_b, t_a)}\right)^{1/2}\,,$$

where the function f must obey

$$m\frac{\partial^2 f(t, t_a)}{\partial t^2} - 2c(t)f(t, t_a) = 0\,, \qquad f(t_a, t_a) = 0\,, \; \left.\frac{\partial f}{\partial t}\right|_{t=t_a} = 1\,.$$

(See L.S. Schulman, *Techniques and applications of path integration*, Chapter 6, for more details.)

(h) Check the above result using the free particle case.

3. The Legendre polynomials are orthogonal functions on the interval $(-1, 1)$ that can be constructed applying the Gram-Schmidt procedure to the set $\{x^\alpha\}_{\alpha=0,1,2,\ldots}$. Construct explicitly the first three *normalized* Legendre polynomials.

4. Repeat the procedure of exercise 3 in the case of the Hermite polynomials, which are orthonormal functions on $(-\infty, +\infty)$ constructed out of $\{e^{-x}x^\alpha\}_{\alpha=0,1,2,\ldots}$

2 More exercises on perturbation theory

1. In non degenerate time-independent perturbation theory, what is the probability of finding in a perturbed energy eigenstate $|E_i\rangle$ the corresponding unperturbed eigenstate $|E_i^{(0)}\rangle$? Solve this up to the second order.

2. A system with three unperturbed states can be represented by the perturbed Hamiltonian matrix

$$\mathcal{H} = \begin{pmatrix} E_1 & 0 & a \\ 0 & E_1 & b \\ a & b & E_2 \end{pmatrix},$$

   where $E_2 > E_1$. The quantities a and b are real numbers, and are perturbations of the same order and much smaller than $E_2 - E_1$. Find the exact eigenvalues. Use the second order degenerate perturbation theory to find the eigenvalues. Compare the results.

3. Find eigenvalues and eigenvectors in degenerate perturbation theory for the Hamiltonian

$$\mathcal{H} = \begin{pmatrix} 0 & \sqrt{2}V_0 & 2\sqrt{2}V_0 \\ \sqrt{2}V_0 & E_0 & 2V_0 \\ 2\sqrt{2}V_0 & 2V_0 & E_0 \end{pmatrix},$$

   defined in the basis $|1\rangle$, $|2\rangle$ and $|3\rangle$.