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1 Exercises

1. Compute the general expressions for the second order corrections to the eigenvalues and eigenvectors in time-independent perturbation theory.

2. Compute the first order correction in time-independent perturbation theory for the Hamiltonian

$$\mathbb{H} = \begin{pmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \delta E \\ 0 & 0 & \delta E \\ \delta E & \delta E & 0 \end{pmatrix} , \tag{1}$$

   where the second term is the perturbation.

3. Consider the Hamiltonian operator

$$\mathbb{H} = \begin{pmatrix} E_0 & \lambda A \\ \lambda A & E_0 \end{pmatrix} , \qquad \lambda \ll 1 \, . \tag{2}$$

   (a) Solve exactly the eigenvalue problem;

   (b) Considering now $\lambda \ll 1$ as a perturbation, apply perturbation theory to compute the eigenvalues and eigenvectors up to $\mathcal{O}(\lambda)$;

   (c) Compare the results of the previous point to what can be obtained Taylor expanding the results of the first question.

4. Apply perturbation theory to the Hamiltonian operator

$$\mathbb{H} = \begin{pmatrix} E_0 & \lambda A & 0 \\ \lambda A & E_0 & \lambda A \\ 0 & \lambda A & E_0 \end{pmatrix} , \qquad \lambda \ll 1 \, , \tag{3}$$

   and compute the first order corrections to the eigenvalues and the eigenvectors.

2 Problems

5. Suppose that the Hamiltonian of a two-level system is given by

$$\mathcal{H}(t) = \begin{pmatrix} E_0 + \epsilon\cos\omega t & -A \\ -A & E_0 - \epsilon\cos\omega t \end{pmatrix}$$

which can be interpreted as an ammonia molecule placed in an oscillating electric field proportional to ϵ that produces an energy shift between the states.

(a) Write the general time dependent state vector $|\psi(t)\rangle$ as a function of the stationary states $|\chi_+\rangle$ and $|\chi_-\rangle$.

(b) Compute the exact evolution of the system. Is it possible to solve these coupled differential equations analytically?

(c) Solve the previously obtained equation with time dependent perturbation theory at first order in ϵ for the case in which system is in the state $|\chi_-\rangle$ of energy $(E_0 + A)$ at a time $t = 0$. Calculate the probability to find the molecule in the state $|\chi_+\rangle$ and $|\chi_-\rangle$ at time t. Show that the probability never exceeds 1.

(d) Obtain an approximate solution in the *rotating-wave or quasi resonant approximation*, namely assuming a weak perturbation due to the electric field $\epsilon \ll A$ and $\omega \simeq 2A/\hbar$. Write everything as a function of $\omega_0 = 2A/\hbar$ and $\omega_1 = \epsilon/\hbar$.

(e) Take $\omega = \omega_0$ and assume that at $t = 0$ the ammonia molecule is in the state $|\chi_-\rangle$ of energy $(E_0 + A)$. Calculate the probability to find the molecule in the state $|\chi_+\rangle$ and $|\chi_-\rangle$ at time t. Show that the probability never exceeds 1.

(f) Show that the result of point (e) reduces to the one of point (c) when the perturbation is small and state precisely what small means in this context as a constraint on $d\mathcal{E}_0$.

(g) At what time does the system first return to its initial state?

(h) Suppose the molecule is in the state $|\chi_+\rangle$ at $t = 0$ and now $\omega \simeq \omega_0$ but $\omega \neq \omega_0$. Calculate the probability of finding the molecule in the state $|\chi_-\rangle$ at time t.

6. Consider an electron at rest in a rotating magnetic field

$$\boldsymbol{B} = B_0\left[\cos(\omega t+\phi)\boldsymbol{e}_x + \sin(\omega t+\phi)\boldsymbol{e}_y\right]$$

where B_0 and ω are constants.

(a) Construct the Hamiltonian for the system in the basis of the z axis.

(b) If the electron (spin 1/2 particle of magnetic moment μ) starts in the spin up state with respect to the z-axis at time $t = 0$, determine the state $|\chi(t)\rangle$ at any subsequent time.

(c) Find the probability of getting $-\hbar/2$ if you measure S_z.

(d) What is the minimum field magnitude B_0 required to force a complete flip in S_z?

7. Alice and Bob are two 30th century lovers. In their time, for some reason, people express their love for one another quantum mechanically. They send electrons to each other through a tiny little box with two openings, corresponding to two orthogonal quantum states, $|A\rangle$ on Alice's side and $|B\rangle$ on Bob's side. The box is able to transfer the electrons from one side to the other by means of the Hamiltonian

$$\mathbb{H} = \alpha(|A\rangle\langle B| + |B\rangle\langle A|). \tag{4}$$

Suppose that, at $t = 0$, Alice puts an electron at her side of the box, and it was empty before that.

(a) What's the probability that Bob finds the electron at his side of the box after a *small* time t? To not have to deal with philosophical implications of entanglement and perspective, suppose that Alice and Bob are in reality a single collective gestalt consciousness that records the results of measurements at the same time for both of them.

They lived happily for a very long time, swapping electrons with each other. Receiving them some times, but on other times not. Just like a bad teenage love story.

Every story needs some conflict, however. Bob eventually had to move away, but he didn't want to abandon his electron-swapping love Alice. As such, they had a plan: connect their boxes to the Quantum Internet!

What is that, you ask? It's a whole new way to send electrons to your loved ones! It establishes an intermediate connection between you, $|A\rangle$ and your lover $|B\rangle$ through an intermediate, *highly energetic* box $|I\rangle$. The Hamiltonian of the Quantum Internet channel between Alice and Bob is

$$\mathbb{H} = \epsilon |I\rangle \langle I| + \gamma(|A\rangle \langle I| + |I\rangle \langle A|) + \gamma(|B\rangle \langle I| + |I\rangle \langle B|), \qquad (5)$$

Where the internet channel is highly energetic, $\epsilon \gg \gamma$. Suppose the initial state of the electron is like in the previous problem.

(b) Using second order perturbation theory on γ (second order in the energies and first order in the eigenstates), find the probability that Bob finds the electron at his end of the box at a small time t to leading order in γ.

(c) Comparing the answers of (a) and (b), what internet speed plan (coupling constant γ) should they purchase for this connection given that they want to maintain the same dynamics of their love life? If the Quantum Internet is to work as advertised, is this consistent with the condition $\gamma \ll \epsilon$?