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## 1 Exercises

1. In the Heisenberg picture, show that the time-dependent observable $\mathbb{A}(t) = \mathbb{U}(t,t_0)^\dagger \mathbb{A} \mathbb{U}(t,t_0)$ satisfies the differential equation $i\hbar\partial_t \mathbb{A}(t) = [\mathbb{A}(t), \mathbb{H}]$;

2. Show explicitly that the basis vector of some hermitian observable in the Heisenberg picture depend on time;

3. Suppose a system is described by two observables

$$\mathbb{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbb{B} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1}$$

   (a) What are the possible outcome values for a measurement of $\mathbb{A}$ and $\mathbb{B}$?

   (b) Are these compatible observables? Write the appropriate Heisenberg inequality.

4. Show that if $\Delta_\varphi A = 0$ then $|\varphi\rangle$ is an eigenvector of the observable $\mathbb{A}$.

5. Show explicitly that $\frac{d}{dt}\langle\psi(t)\,|\,\psi(t)\rangle = 0$ requires a self-adjoint Hamiltonian.

6. Can the vectors $|\chi\rangle = (a, b)$ and $|\varphi\rangle = (|a|, |b|e^{i\gamma})$ describe the same state? In which cases (if any) this is true?

7. Suppose we lived in a universe in which the Schrödinger equation contains second order time derivatives, $i\hbar\partial_t^2 |\varphi(t)\rangle = \mathbb{H}\,|\varphi(t)\rangle$. Would it be true that the norm of $|\varphi(t)\rangle$ is time independent?

8. A stationary spin 1/2 particle of magnetic moment $\boldsymbol{\mu}$ in a magnetic field $\boldsymbol{B}$ can be described quantum mechanically as a two-level system by using the magnetic moment as the operator

$$\boldsymbol{\mu} = \mu\boldsymbol{\sigma}. \tag{2}$$

where the $\boldsymbol{\sigma}$ matrices (Pauli matrices) are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

We can then construct the Hamiltonian for this system by analogy with the classical system,

$$\mathbb{H} = -\boldsymbol{\mu} \cdot \boldsymbol{B}. \tag{4}$$

Compute the time evolution operator $\mathbb{U}(t)$ in the basis induced by the Pauli matrices.

9. Consider the most general two-level time-dependent Hamiltonian

$$\mathbb{H}(t) = \begin{pmatrix} a(t) & b(t) \\ b^*(t) & c(t) \end{pmatrix}, \tag{5}$$

where $a(t)$ and $c(t)$ are real-valued functions.

   (a) Find conditions on the three functions such that the Hamiltonian commutes with itself for all times, i.e.

$$[\mathbb{H}(t), \mathbb{H}(t')] = 0, \, \forall t, t' \tag{6}$$

   (b) Use these conditions to write $b(t)$ in terms of $a(t)$, $c(t)$ and a single, constant complex number $z$.

   (c) Compute the time evolution operator for this simplified Hamiltonian. (Hint: write it in terms of Pauli Matrices and follow the strategy of the last problem)

# 2 Problems

10. We have seen that the Hamiltonian for the ammonia molecule in the $\{|1\rangle, |2\rangle\}$ basis is

$$\mathbb{H} = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}, \tag{7}$$

which has eigenvectors $\{|E_0 - A\rangle, |E_0 + A\rangle\}$.

(a) Check explicitly that $\{|E_0 - A\rangle, |E_0 + A\rangle\}$ is an orthonormal basis in $\mathbb{C}^2$;

(b) Compute the matrix of the change of basis between $\{|1\rangle, |2\rangle\}$ and $\{|E_0 - A\rangle, |E_0 + A\rangle\}$. Is it unitary as expected?

(c) Since the matrix of the change of basis between $\{|1\rangle, |2\rangle\}$ and $\{|E_0 - A\rangle, |E_0 + A\rangle\}$ is real, parametrize it as a rotation matrix, properly defining the rotation angle;

(d) Write the Hamiltonian operator in the $\{|E_0 - A\rangle, |E_0 + A\rangle\}$ basis;

(e) Compute the time evolution operator in the $\{|E_0 - A\rangle, |E_0 + A\rangle\}$ basis.

(f) Suppose we prepare the ammonia molecule in the ground state $|E_0 - A\rangle$ at $t_0 = 0$. Compute the probability to find the system in the excited state after a time $t$.

(g) If we were to put the molecule in a small constant electric field $\boldsymbol{\mathcal{E}}$ pointing from site $|1\rangle$ to $|2\rangle$ we expect the Hamiltonian to be mostly unchanged, except for a small shift in energy $q\mathcal{E}d$, where $q$ is an effective charge and $d$ is the distance between the electronic sites. If we start with the molecule on site $|1\rangle$ at $t_0 = 0$, compute the probability for it to be measured in site $|2\rangle$ for $t > t_0$.

11. In 1947, C. Butler and G. Rochester discovered the first particle containing the strange quark, dubbed neutral $K$ meson. In terms of quark, there are two possibilities for the constituent quarks: we either have a bound state composed by a strange and an anti-down quark $s\bar{d}$ (denoted by $\overline{K^0}$), or another bound state composed by an anti-strange and a down quark $\bar{s}d$ (denoted by $K^0$). The Standard Model of particle physics predicts that there is a non-vanishing probability for $\overline{K^0}$ to turn into $K^0$ and viceversa. Does the system undergo oscillation? Motivate the answer.

12. The sun can be considered a "neutrino factory", since a huge neutrino flux of $6.5 \times 10^{14}\,\nu/\mathrm{s}$ reaches Earth. The main nuclear reaction for $\nu_e$ production is $p^+ + p^+ \to {}^2\mathrm{H}^+ + e^+ + \nu_e$, where $\nu_i$ indicates a neutrino of flavor $i$. Since the physics of the interior of the Sun is very well known since the 1960's (it is actually much better known than the physics of the interior of the Earth), it is possible to predict the solar neutrino flux on Earth.

In 1964 the Homestack experiment was developed to measure the electron neutrino flux on Earth, measuring about 1/3 of the expected number of $\nu_e$. The result persisted since 2001, when the SNO experiment measured both the $\nu_e$ and the total $\nu_e + \nu_\mu + \nu_\tau$ flux, confirming the Homestack result and observing that the total number of neutrinos reaching Earth matched the $\nu_e$ flux produced in the sun. This phenomenon is called *neutrino oscillation*, and has been experimentally confirmed with other neutrino sources (atmospheric and reactor).

The explanation for the oscillation is quantum mechanical, in analogy to the oscillation of the ammonia molecule. Although the neutrino system is a three level system ($\nu_e + \nu_\mu + \nu_\tau$), we can in first approximation consider it as a two level system involving $\nu_e$ and $\nu_\mu$ only. Suppose we have some mystery neutrino moving in a certain direction with very high momentum $p$.

(a) Choose appropriately the $\{|\nu_e\rangle, |\nu_\mu\rangle\}$ basis. Explain what these states represent.

(b) What must be the form of the Hamiltonian in the $\{|\nu_e\rangle, |\nu_\mu\rangle\}$ basis for the flavor to have oscillations?

(c) Compute the states of definite mass using the ultrarelativistic approximation,

$$E_i = \sqrt{(pc)^2 + (m_i c^2)^2} \approx pc + \frac{m_i^2 c^3}{2p}. \tag{8}$$

Write the original Hamiltonian in the flavor basis in terms of the two definite masses $m_+ > m_-$.

(d) Consider now that an electron neutrino is created in the sun at $t_0 = 0$. What is the time evolution of the state in the flavor basis?

(e) What is the probability of measuring a $\nu_e$ neutrino at the time $t$? Write it in terms of the ultrarelativistic energy $E = pc$. This is called the *survival probability*.

(f) Suppose that the Sun emits a fixed flux $\Phi$ of electron neutrinos per unit time. Given that the Earth is very far away from the Sun, show that our measured flux of electron neutrinos is only $\Phi/2$.

13. In this problem we are going to study how the motion of a free quantum particle in one spatial dimension arises from finite dimensional QM,

using some intuitive reasoning based on classical mechanics. Inspired by the benzene molecule, consider a closed chain of $N$ identical atoms with a single energy level $\epsilon$. We put an electron in that chain that can hop to the nearest neighbors, through the Hamiltonian

$$\mathbb{H} = \sum_{j=1}^{N} \epsilon \left|j\right\rangle \left\langle j\right| - \gamma \left(\left|j\right\rangle \left\langle j+1\right| + \left|j\right\rangle \left\langle j-1\right|\right) \tag{9}$$

where orthogonal basis of kets $\{|j\rangle\}$ indicate that the electron is mostly localized around atom $j$. Since the chain is supposed to be closed, we also identify $|j\rangle \equiv |N+j\rangle$. The choice of sign for the hopping parameter $\gamma$ will become clear later.

(a) Show that $\mathbb{H}$ is self-adjoint.

To see how this Hamiltonian can describe the motion of the electron, define a position operator

$$\mathbb{X} = \sum_{j=1}^{N} x_j \left|j\right\rangle \left\langle j\right| \tag{10}$$

where $x_j$ corresponds to the position of atom $j$ along the chain. We then take, for all $j$, $x_j - x_{j-1} = a$, the separation between sites.

We now define a velocity operator $\mathbb{V}$ in the following way. For any time evolved state, we must have

$$\langle \mathbb{V} \rangle = \frac{d \langle \mathbb{X} \rangle}{dt}. \tag{11}$$

(b) Write down an expression for $\mathbb{V}$ in terms of $\mathbb{H}$ and $\mathbb{X}$. Then write down $\mathbb{V}$ explicitly.

(c) Show that $[\mathbb{H}, \mathbb{V}] = 0$.

This last property guarantees that velocity eigenstates are energy eigenstates, and so we can analyze the former more thoroughly in order to solve the system.

(d) Show that

$$|\phi\rangle_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{ikj} |j\rangle \tag{12}$$

is a velocity eigenstate. What are the allowed values for $k$? Show that if $N$ is very large (obcenely large, really), there's a large amount of states whose eigenvalues $v$ are related to $k$ by

$$k \approx \frac{\hbar v}{2\gamma a}. \tag{13}$$

(e) What is the eigenvalue of $\mathbb{H}$ is associated to $|\phi_k\rangle$? Suppose we want to describe the dynamics of a particle of mass $m$. In the very large $N$ approximation, what choices of $\epsilon$ and $\gamma$ give rise to the kinetic energy of classical mechanics? Namely,

$$E = \frac{1}{2} m v^2. \tag{14}$$

(f) Using those choices of $\epsilon$ and $\gamma$, define the momentum operator

$$\mathbb{P} = m\mathbb{V}. \tag{15}$$

Show that

$$[\mathbb{X}, \mathbb{P}] = i\hbar \sum_{j=1}^{N} \frac{|j\rangle \langle j+1| + |j\rangle \langle j-1|}{2} \tag{16}$$

Note how this is very similar to the canonical commutation relation, $[\mathbb{X}, \mathbb{P}] = i\hbar$, just that the terms are completely off-diagonal. Since, however, we're eventually going to take a very large $N$ approximation, it makes sense that we would not be able to distinguish between positions so close together such as $j$ and $j+1$. Approximating these neighbors to be very similar, we get the canonical commutation relation.

$$[\mathbb{X}, \mathbb{P}] = i\hbar \tag{17}$$