List of exercises 10

1. The wave function of a particle subject to a spherically symmetrical potential V(r) is given by

$$\psi(\vec{x}) = (x + y + 3z) f(r).$$

   (a) Is ψ an eigenfunction of $\hat{L}^2$? If so, what is the ℓ-value? If not, what are the possible values of ℓ we may obtain when $\hat{L}^2$ is measured?

   (b) What are the probabilities for the particle to be found in various ℓ_z states?

   (c) Suppose it is known somehow that $\psi(\vec{x})$ is an energy eigenfunction with eigenvalues E. Indicate how we may find $V(r)$.

2. **Wigner-Eckart 1:** consider a spin-less particle bound to a fixed center by a central force potential.

   (a) Relate, as much as possible, the matrix elements

$$\langle n', \ell', m'| \mp \frac{1}{\sqrt{2}}(x \pm iy)|n, \ell, m\rangle \qquad\qquad \langle n', \ell', m'|z|n, \ell, m\rangle$$

   using *only* the Wigner-Eckart theorem. Discuss under what conditions the matrix elements are non-vanishing.

   (b) Do the same using the wave functions $\psi(\vec{x}) = R_{n\ell}(r) Y_\ell^m(\theta, \phi)$.

3. **Wigner-Eckart 2:**

   (a) Write xy, xz and $(x^2 - y^2)$ in terms of a components of a spherical (irreducible) tensor of rank 2.

   (b) The expectation value

$$Q \equiv e\langle \alpha, j, m = j|(3z^2 - r^2)|\alpha, j, m = j\rangle$$

   is know as the quadrupole moment. Evaluate

$$e\langle \alpha, j, m = j|(x^2 - y^2)|\alpha, j, m = j\rangle,$$

   where $m' = j, j-1, j-2, ...$ in terms of Q and appropriate Clebsch-Gordan coefficients.

4. **Wigner-Eckart 3:** Let us label as $|n, \ell, m\rangle$ the eigenstates of the hydrogen atom. Let χ be

$$\chi = \langle n = 3, \ell = 2, m = 2|xy|n = 3, \ell = 0, m = 0\rangle.$$

Compute as a function of χ,

$$\chi = \langle n = 3, \ell = 2, m|\mathcal{O}_j|n = 3, \ell = 0, m' = 0\rangle,$$

where $\mathcal{O}_j = xy, xz, yz, xx, yy, zz$.

5. **Wigner-Eckart 4:** Let $|j = 3/2, j_z\rangle$ label the simultaneous eigenvectors of the angular momentum operators J^2 and J_z. Evaluate

$$\langle j = 3/2, j_z'|J_m|j = 3/2, j_z\rangle, \qquad m = x, y, z$$

and show that the results are in agreement with the Wigner Eckart theorem.

6. **Fine structure of hydrogen:**

(a) The kinetic energy of a particle of rest mass m and momentum p according to special relativity is

$$K = \sqrt{p^2c^2 + m^2c^4} - m.$$

Compute its non relativistic limit ($p \ll mc$) and compare the result with the standard non relativistic expression for the kinetic energy.

(b) Compute the effect of this type of correction on the energy levels of a hydrogen atom. Does the new term commute with $\hat{L}^2$ and $\hat{L}_z$? Use the correct (degenerate or non-degenerate) first order perturbation theory.

(c) The electron in a hydrogen atom experiences an electric field $\vec{E} = \frac{e\hat{r}}{4\pi\epsilon_0 r^3}$ due to the charge of the nucleus. Furthermore, a non-relativistic particle moving in an electric field with velocity $\vec{v}$ experience an effective magnetic field $\vec{B} = -\frac{\vec{v}\times\vec{E}}{c^2}$. Built the additional contribution to the Hamiltonian due to this effect. (*Hint:* remember that an electron posses a magnetic moment $\mu = -e/m_e\hat{S}$).

(d) Apply perturbation theory to the hydrogen atom using the new perturbed Hamiltonian and compare with the result of point (b). Are the two results of the same order? Can you add these corrections together?

(e) Draw the effect(s) find in the previous points on the $n = 1, 2$ and 3 energy states of a hydrogen atom.

7. **Hyperfine structure of hydrogen:** the proton inside the hydrogen atom posses a magnetic moment

$$\mu_p = \frac{g_p e}{2m_p}\hat{S}_p,$$

where S_p is the proton spine and $g_p = 5.59$ is the proton gyromagnetic ratio. The proton's magnetic moment generates the following magnetic field

$$\vec{B} = \frac{\mu_0}{4\pi r^3}\left[3(\vec{\mu}_p \cdot \vec{e}_r)\vec{e}_r - \vec{\mu}_p)\right] + \frac{2\mu_0}{3}\vec{\mu}_p\delta^3(\vec{r}),$$

where $\vec{e}_r = \vec{r}/r$. The Hamiltonian of the electron in the magnetic field generated by the proton is $H_1 = -\vec{\mu}_e \cdot \vec{B}$ with $\vec{\mu}_e = -e/m_e\vec{S}_e$.

(a) Treating H_1 as a perturbation, find the corrections to the hydrogen atom eigenvalues induced by this new term for a generic state.

(b) What is the energy shift for the ground state?

(c) Using the fact that $\vec{S} = \vec{S}_e + \vec{S}_p$ show that the spin-spin coupling break the degeneracy of the two $1S_{1/2}$ states of hydrogen. This lifting is know as *hyperfine structure*.

(d) Is the hyperfine energy shift larger or smaller than the fine structure energy shift? By how much?