List of exercises 1

1. Consider the matrix
$$\mathbb{A} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$
Compute $\mathbb{A}^\dagger$, $\mathrm{tr}(\mathbb{A})$ and $\det(\mathbb{A})$. Is this matrix hermitian? Is it unitary?

2. Verify that the operator $P_\varphi = |\varphi\rangle\langle\varphi| / \|\varphi\|^2$ satisfies all the properties of projectors.

3. Prove that the trace and the determinant of an operator are independent from the choice of basis.

4. Consider the two vectors $|a_1\rangle = (1, 1)$ and $|a_2\rangle = (2, 1)$:
   (a) Show that the two vectors are linearly independent;
   (b) Apply the Gram-Schmidt orthonormalization procedure to construct an orthonormal basis starting from $\{|a_1\rangle, |a_2\rangle\}$;
   (c) Compute the basis that connects the just found orthonormal basis with the canonical basis $\{|e_1\rangle = (1, 0), |e_2\rangle = (0, 1)\}$ and check its unitarity;
   (d) Verify that the basis constructed satisfies the resolution of the identity;
   (e) Is it true that the set of vectors $\{|a_1\rangle, |a_2\rangle\}$ themselves satisfies the resolution of the identity?

5. Prove that for an operator $\mathbb{A}$ with eigenvalues $\{a_1, \ldots, a_n\}$ we have $\mathrm{tr}\mathbb{A} = \sum_k a_k$ and $\det\mathbb{A} = \prod_k a_k$;

6. Given the matrix
$$\mathbb{T} = \begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix};$$
   a) compute the eigenvalues $\lambda_{1,2,3}$;
   b) compute $\mathrm{tr}(\mathbb{T})$ and $\det(\mathbb{T})$;
   c) show explicitly that $\mathrm{tr}(\mathbb{T}) = \lambda_1 + \lambda_2 + \lambda_3$ and $\det(\mathbb{T}) = \lambda_1\lambda_2\lambda_3$;

d) compute the eigenvectors;

e) write, if possible, the spectral decomposition of $\mathbb{T}$;

f) compute $e^{\mathbb{T}}$;

7. The Pauli matrices are defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) Show that $\sigma_i^\dagger = \sigma_i$, $i = 1, 2, 3$;

b) Show that $\sigma_i \sigma_j = \delta_{ij} + \epsilon_{ijk}\sigma_k$;

c) Deduce that $(\vec{a}^T \cdot \vec{\sigma})(\vec{b}^T \cdot \vec{\sigma}) = (\vec{a}^T \cdot \vec{b}) + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma}$, where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T$;

d) Compute eigenvalues and eigenvectors of each matrix σ_i;

e) Compute $e^{i\vec{a}\cdot\vec{\sigma}}$.

8. Let M be a matrix in $\mathbb{C}^2$.

a) Is it true that $M = c1 + \vec{a}^T \cdot \vec{\sigma}$ with $(a_i, c) \in \mathbb{C}$?

b) Using the properties of the $\vec{\sigma}$ matrizes, compute the coefficients c e $\vec{a}$;

c) With which product the basis $\{1, \vec{\sigma}\}$ is orthogonal?

d) Compute e^M;

e) Decompose M^{-1} in the $\{1, \vec{\sigma}\}$ basis.

9. Following the structure of the Pauli matrices, write all the 3×3 hermitian matrices with vanishing trace. These 8 matrices are called Gell-Mann matrices λ_a, $a = 1, \ldots, 8$. Is it true that the set $\{1, \lambda_{1,\ldots,8}\}$ is a basis for the matrix space $\mathbb{C}^3$?